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Contributions to the investigations of Lascar strong types in simple theories

por

Steffen Lewitzka

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Orientador: Prof. Ruy José Guerra B. de Queiroz

Supervisor: Prof. Ruy José Guerra B. de Queiroz

Contents

Abstract

First, we shall prove some results about thick formulas and bounded equivalence relations, valid in arbitrary complete theories. The application of these results to simple theories yields some nice and useful properties with respect to Lascar strong types.

Furthermore, we consider subclasses of simple theories, definable by dividing chains as in [CasWag]. As one of our main results we find a considerable improvement of a Theorem due to Kim (Proposition 3.6 in [Kim1], or Theorem 2.4.7.6 in [Wag]), which describes dividing by means of Morley sequences. This leads us to new and promising characterizations of the class of low theories. We study simple theories having a special property that we call the independent dividing chain property. Another main result is that simple, ω-categorical theories having this property are low. We define and study a new rank that allows characterizing short and low theories and give rise to further studies. Furthermore, we develop a rapprochement to the investigation of the equality of strong types and Lascar strong types using the preceding results about bounded equivalence relations. Finally, in the last section we develop and outline some promising ideas for further studies in simple theories proceeding from the main results of this thesis. These considerations could serve in future works to tackle open questions such as the Lstp=stp problem (equality of Lascar strong types and strong types) and may lead to a better understanding of the relationships between some subclasses of simple theories.

Introduction

This thesis is divided in two parts. The first one is less formal and is dedicated to give an overview about some studies of Finite Model Theory [EbbFl] and a motivation to investigate some areas of classical (infinite) Model Theory, from the point of view of Theoretical Computer Science [Imm]. We shall show in which way Finite Model Theory can serve as a powerful tool to handle problems of Theoretical Computer Science, in particular Complexity Theory. Some main results and open problems of Descriptive Complexity – the discipline connecting Complexity Theory and Finite Model Theory and Logic, – will be discussed. Furthermore, we go into a topical development of Model Theory, which deals with finite structures using methods of classical Model Theory, especially Stability Theory and the sophisticated machinery developed by Shelah. This development, called Embedded Finite Model Theory, seems to be very promising with respect to a new approach to open problems in Descriptive Complexity.

The real objects of our investigation and all results obtained by the author are developed in the second part of the thesis. In the first seven chapters of the second part we shall give a formal and detailed introduction into simple first order theories, necessary for the understanding of the following matters and our main results. In particular, we treat non-forking dependence (as developed by Shelah) which gives rise to an independence notion between the sets of a model. Most of this material represents the results due to Kim and Pillay [Kim1], Shelah [Sh1] and Wagner [Wag]. Our presentation of these subjects follows largely [Wag].

Simple theories contain the stable theories but also many more, like the theory of the random graph and the theory of some important structures of Embedded Finite Model Theory (namely the smoothly approximable structures). This fact justifies our interest in simple theories. Moreover, it seems that stable theories are going to be replaced by simple theories in many foundational investigations of pure Model Theory.

Our main interest in simple theories is some amalgamation properties about types over sets. Two non-forking extensions of any type over a model can be amalgamated by a common non-forking extension. This is called the Independence Theorem over a model. In stable theories, the Independence Theorem over a model or over algebraically closed sets turns out to be trivial, since all types over models and algebraically closed sets have exactly one non-forking extension to any superset. When Kim and Pillay studied this matter in the general context of simple theories [Kim1], they discovered that the concept of strong type (types over algebraically closed sets) must be extended to the notion of Lascar strong type in order to prove a similar amalgamation property in this broader class of theories, called the Independence Theorem for Lascar strong types. This theorem says that (in simple theories) two non-forking extensions of a Lascar strong type have a common non-forking extension. Therefore, a Lascar strong type is also called an amalgamation base. These subjects will be discussed in the chapters 2.6 and 2.7.

Amalgamation properties of types as given in the different Independence Theorems are some of the core subjects in investigation of model theory. "Type amalgamation … is perhaps the most useful property of forking dependence in a simple theory." (Buechler [Bue2]). It is interesting to ask whether Lascar strong types are equivalent to strong types in simple theories. The equivalence of Lascar strong types and strong types gives additional information on the non-forking extensions of a type. For instance, if A is an algebraically closed set, then any two non-forking extensions of a complete type over A have a common non-forking extension. This follows from the Independence Theorem for Lascar strong types and the equivalence of Lascar strong types and strong types. The question whether this equivalence holds is called the Lstp=stp problem and is answered positively for some subclasses of simple theories, but the answer is not known for simple theories in general.

Apart from Lascar strong types, there is another concept, which emerges in the study of simple theories (and has no significance in stable ones), namely hyperimaginary elements. These objects are classes of type-definable equivalence relations and appear as canonical bases of types in simple theories. In this thesis, we are not too much interested in the theory of canonical bases (see [HKP] for a further study), however, we will see that the elimination of hyperimaginaries (that is, their equivalence to sequences of imaginary elements) is a condition which implies Lstp=stp.

For these reasons, we shall study the matters regarding strong types and Lascar strong types and expose some basic theory of hyperimaginaries. The question whether simple theories eliminate hyperimaginaries, and the (weaker) Lstp=stp problem are the main open problems in the study of simple theories. A stable theory eliminates hyperimaginaries. In [BPW] this is shown for the subclass of supersimple theories. Buechler [Bue2] was able to show that in another subclass, the low theories, Lstp=stp holds.

In chapter 2.8 we study certain bounded equivalence relations in an arbitrary theory and their relationships to Lascar strong types. Then we investigate these results under the assumption that the ambient theory is simple and obtain some nice properties. The subjects of this chapter were developed in detail during a short stay of the author of this thesis at the University of Barcelona (Spain) in 2000 under the supervision of Prof. Enrique Casanovas [Cas3]. In particular, the results 2.8.34 – 2.8.39 and 2.8.44 are due to the author (obtained in 2001); a new and shorter proof of 2.8.43 is following from these results.

Chapter 2.9 is dedicated to the basic theory of hyperimaginaries in simple theories. A simple theory which eliminates hyperimaginaries satisfies Lstp=stp. However, elimination of hyperimaginaries is only proved for the subclass of supersimple theories [BPW] and seems to be a very difficult problem for the general case. The author developed the proofs of Propositions 2.9.14, 2.9.16 (together with Prof. Casanovas), and 2.9.21. Lemma 2.9.16 gives an important characterization of Lascar strong types in terms of certain bounded equivalence relations under the assumption of simplicity of the theory. The proof given herein (due to the author of the thesis) of 2.9.19 is an improvement and simplification of a proof due to Casanovas [Cas3].

Now, in chapter 2.10, we are able to define and characterize the Lstp=stp problem. The results 2.10.6 –2.10.9 were developed in the above-mentioned stay at the University of Barcelona with the participation of the present author.

Finally, in chapter 2.11, we will present our main results. Before that, we define some subclasses of simple theories, starting with the stable ones which have very nice properties with respect to amalgamation, non-forking extensions of types, definability of types and the elimination of hyperimaginaries. The definitions of further subclasses of simple theories, such as the low, superlow, supershort and short theories, by means of dividing chains, are due to Casanovas [CasWag]. However, the low theories was defined (by means of a rank) and investigated before by Buechler [Bue2]. He proved that Lstp=stp holds in low theories. All known natural examples of simple theories are low; in particular the smoothly approximable structures are low. It is not known whether low theories or (super-) short theories eliminate hyperimaginaries and the Lstp=stp problem still remains open in (super-) short theories.

However, the author of this thesis was able to give new characterizations of low theories by means of Morley sequences. These characterizations could serve as useful tools to identify other simple theories with the low theories, for instance ω-categorical simple theories having additional properties. Casanovas in [CasWag] proved that ω-categorical short theories are low. We discovered that a special property, which we introduce as the "independent dividing chain property", is a sufficient condition for lowness of a ω-categorical simple theory. This is one of our main results. At the heart of the proof is another main result of this thesis, namely an improvement and generalization of Kim's early Theorem (Proposition 3.6 in [Kim1] or Theorem 2.4.7.6 in [Wag]) that if a formula φ(x,a) divides over a set A, then for every Morley sequence I of tp(a/A), the set {φ(x,a') : a'∈I} is inconsistent. Kim's proof does not provide a number k, such that {φ(x,a'): a'∈} is k-inconsistent. We were able to find such a k and show that I is an m-dividing chain in φ, if φ(x,a) m-divides over A.

Furthermore, a new rank is defined, which allows characterizations of low and short theories and gives rise to research questions. Finally, a new rapprochement to the Lstp=stp problem is developed by the author, using results of chapter 2.8. The main results of these considerations are given in Propositions 2.11.38 and 2.11.39. Roughly speaking, they say that to show Lstp=stp in a simple theory, it is sufficient to prove that certain definable equivalence relations are finite. All results 2.11.29 – 2.11.44 are due to the author of this thesis.

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1 From Theoretical Computer Science to Model Theory

The notion of algorithm is the central concept in Theoretical Computer Science. An algorithm is – roughly speaking – an effective procedure, that is, a finite set of instructions to transform input data into output data in finitely many steps. We require that all data are finite objects, or more precisely, that they are finite words over a finite alphabet. Hence, we can restrict our attention to such countable domains like the natural numbers or finite words over, say, {0,1} (there exists an injective function coding all data into, for instance, natural numbers).
In Recursion Theory – one of the main pillars of Theoretical Computer Science – one tries to formalize the intuitive understanding of the notion of algorithm.
We say that a function f: $\mathbf{N}^k \rightarrow \mathbf{N}$, ($k \in \mathbf{N}$, **N** the natural numbers), is intuitively computable (or intuitively recursive), if there is an algorithm (in the sense described above) which computes f, which means that there is a procedure which takes $(n_1,...,n_k) \in \mathbf{N}^k$ as input and stops after finitely many steps with $f(n_1,...,n_k) \in \mathbf{N}$ as output. (If f is partial, then we require that the algorithm does not stop on some arguments of f.) (Clearly, all that computers can do is computable (recursive) in the intuitive sense.)
In the 1930's various models of computability were developed in order to treat the intuitive notion by formal methods. Some of them serve even today to study the notion and the properties (like complexity in time and space) of algorithms, for example Turing Machines, Markov Algorithms, λ-calculus, μ-recursive functions, Register Machines,....
It was shown that all these models are equivalent, which means that they define exactly the same class of functions over **N**, called recursive functions.
It is clear that all these functions are intuitively computable (intuitively recursive). The equivalence of the formal models of computability and the apparent impossibility to find a stronger formal concept which generates more than the recursive functions and still obeys the intuitive understanding of an algorithm gave rise to the assumption that the class of the intuitively recursive functions and the class of the recursive functions coincide. This led A. Church to state his famous thesis (Church's Thesis, 1936) that identifies the intuitive with the formal notion of computabilty. That is:
A function f over **N** is intuitively recursive if and only if it is recursive.

Church's Thesis is not a theorem of mathematics, since it contains the notion of an effective procedure, which is only intuitively explained. But it is accepted and confirmed in practice until today and provides an important philosophical understanding of computability.

A subset of $\mathbf{N}^k$, ($k \in \mathbf{N}$), is said to be decidable or recursive, if its characteristic function is recursive. Recursion Theory provides a structure theory of the subsets of $\mathbf{N}^k$, namely the Kleene-Hierarchy, dividing these subsets by the complexity of their definitions by first order formulas (number of alternating quantifiers in front of a recursive relation). The bottom of this hierarchy (more exactly the relations which are both recursively enumerable (r.e.) and complements of r.e. relations, that is the relations definable by both formulas with one existential quantifier in front of a recursive relation and formulas with one universal quantifier in front of a recursive relation) is formed by the recursive relations.

So Recursion Theory goes beyond the world of recursiveness. But Computer Science, as the science of computability is restricted to investigate this bottom level of the Kleene Hierarchy, that is the recursive relations or, equivalently, the decidable (computable) problems. While the mathematician is more interested in undecidable problems (like the theory of the natural numbers, the theory of groups, rings, fields, the word problem in group theory, the halting problem of Turing machines, ...) considering decidable (recursive) sets as trivial, the work of the computer scientist begins here in the world of recursiveness. In this framework of the recursive problems there arose Complexity Theory (Computational Complexity), Automata Theory and others.

The existence of an algorithm of a (recursive) problem raises the natural – and, in particular, from the engineering standpoint, important - question about its complexity in time and space required by a machine that carries out the algorithm. (We will see that the question of the complexity of an algorithm is not only interesting for the engineer but has also philosophical and mathematical significance.) It was important to discover that the complexity essential does not depend on the model of computation mentioned above (Turing Machine, Register Machine, μ-recursive function ...). Thus, the notion of complexity in time and space can be considered as a fundamental concept and we can speak about the ¨complexity of the problem¨ without referring to a special machine model or calculus.

This allows to classify the recursive sets into complexity classes corresponding to the amount of resources required by one of the equivalent models of computation to accept the set. Usually one chooses the intuitive very clear model of Turing Machines to work in practice.

Let us explain this classification of the recursive sets more exactly. We code a recursive problem (set) into a decidable language over an alphabet τ, say $\tau=\{0,1\}$. By T(M) we mean the language A over τ which is accepted by the Turing machine M. (The language A is accepted by the Turing machine M, if M stops after finitely many steps on all inputs $a\in A$.)
Let $f:\mathbf{N}\to\mathbf{N}$ be a function. The class TIME(f(n)) consists of all languages A such that there is a Turing machine M with A=T(M) and $time_M(x) \leq f(|x|)$, where $time_M:\tau^*\to N$ means the number of steps of the calculation of M on the input $x\in\tau^*$. If we also admit non deterministic Turing machines, then let $time_M(x)$ be the minimum number of steps of possible calculations of M on input x. |x| denotes the length of x.
The class SPACE(f(n)) is defined in an analogous way replacing $time_M$ by $space_M:\tau^*\to N$, which means the number of cells of the work tape which have been scanned (or the minimum number of the possible numbers of scanned cells in the non deterministic case).

Some important complexity classes are:

PTIME (=P)=$\cup\{$TIME(p(n))|p a polynomial function, all considered Turing machines are deterministic$\}$

NPTIME (=NP) is defined like P, but non-deterministic Turing machines are admitted

PSPACE (=PS)=$\cup\{$SPACE(p(n))| p polynomial, only deterministic Turing machines$\}$

LOGSPACE (=L)=$\cup\{$SPACE(f(n))| f(n)=[clog(n)], c>0, all M deterministic$\}$, where [r] means $\min\{m \mid r\leq m, m\in N\}$, for r a real number

NLOGSPACE (=NL) is the non-deterministic version of L.

Obviously we have $L\subseteq NL\subseteq P\subseteq NP\subseteq PSPACE$.

However, nothing is known about strict inclusion up to $NL\subset PSPACE$.

In this way Computational Complexity provides a fine structure theory of the recursive world inside the Kleene-Hierarchy, dividing the recursive sets into complexity classes. So we can consider Computational Complexity as a ¨sub theory¨ of Recursion Theory.

However, measuring the complexity of a problem by the resources of time and space of a machine model is more an engineering standpoint to understand the phenomenon but does not allow an elegant, general mathematical approach.

Such an approach was discovered in 1974, when Ron Fagin [Fag] showed that the complexity class NP is exactly the set of problems describable in second order existential logic. This was a spectacular breakthrough in Theoretical Computer Science. Since then more complexity classes were identified by appropriate extensions of first order logic and it became clear that the measures time and space are expressible by the richness of a logic language needed to specify the problem. Thus, in this way the apparently technical measures time and space are confirmed as deep mathematical concepts. The area, which investigates the world of computability by identifying natural complexity classes via logic languages, is called Descriptive Complexity and shows that virtually all measures of complexity can be mirrored in logic.

The way to connect complexity classes with appropriate logic languages goes about finite (word-) models (finite structures). The discipline, which deals with the interaction between theories (sets of sentences) of a logic language and their semantics, i.e. an interpretation in structures, is called Model Theory. Below we shall give some general remarks about Model Theory, its methods and some highlights of its history. But first let us return to the question as to why Model Theory becomes an important tool for us.

As mentioned above, we can understand a (recursive) problem coded as a formal language, i.e. as a set of words over, say, $\{0,1\}$. In the following we will give a short survey of how words can be considered as finite structures and, on the other hand, how finite structures can be coded as words. This way formal languages naturally correspond to classes of structures. Now, classes of structures can be classified by their descriptive complexity, that is, by the logical means that are necessary to axiomatize them, i.e. to write them as model classes of sentences of a certain logic.

Vocabularies τ are finite sets consisting of relation symbols R,... and constant symbols c,... . For simplicity we mainly consider the case $\tau=\{E\}$ with a binary relation symbol E. By a τ-structure we mean a finite τ-structure. Str(τ) is the class of all (finite) τ-structures. By a class of structures we mean a class closed under isomorphisms. We shall code structures by words over $\{0,1\}$. For this we need a linear ordering on the domain. Let $\tau^<:=\tau\cup\{<\}$, with < a binary relation symbol, $<\notin\tau$, and $OStr(\tau^<):=\{(A,<^A)|\ A\in Str(\tau),\ <^A$ a linear ordering of A$\}$ the class of ordered structures. Then $<^A$ induces the lexicographic orderings $<^A{}_2$, $<^A{}_3$, ... on A^2, A^3, ... for $(A,<^A)\in OStr(\tau^<)$.

Define, e.g. for $\tau=\{E\}$, $code:OStr(\tau^<)\to\{0,1\}^*$ by $code(A,E^A,<^A):=1...1w_EA$, (|A| times ¨1¨), where $w_EA=e_0...e_{|A|}2\text{-}1$, with $e_i=1$, if $E^A(a,b)$ for the i-th pair $(a,b)\in A^2$ in the ordering $<^A{}_2$; $e_i=$ 0, else.

Elements c^A of A are coded by bin(i), the binary representation of i, where c^A is the i-th element of A with respect to $<^A$. For $K\subseteq OStr(\tau^<)$ let $code(K):=\{code(B)|\ B\in K\}$.

Then we have that code(K) is a formal language over $\{0,1\}$. The following definition allows handling structures without orderings:

Let $\mathbf{M}\subseteq Str(\tau)$, $<\notin\tau$. $\mathbf{M}^<:=\{(A,<^A)\in OStr(\tau^<)|\ A\in\mathbf{M}\}$.

Note that for all $A\in Str(\tau)$:

$A\in\mathbf{M}$ iff there is $<^A$ such that $(A,<^A)\in\mathbf{M}^<$,

iff for all orderings $<^A$ we have $(A,<^A)\in\mathbf{M}^<$.

Let code:Str(τ)$\to$powerset of $\{0,1\}^*$ be defined by:

$code(A):=\{code(A,<^A)|\ <^A$ an ordering on A$\}$, and put $code(\mathbf{M}):=\cup\{code(A)|\ A\in\mathbf{M}\}$.

Thus, with each class of structures, in particular with each class of ordered structures, we have associated a formal language. Concerning the other way round, we code words by ordered $\{P\}$-structures for a unary relation symbol P, i.e. by structures from $OStr(\{P\}^<)$.

For this purpose, let $w \in \{0,1\}^*$ be of length n. Set $B_w := (B, P^B, <^B)$ with $B := \{0,...,n-1\}$, $<^B$ the natural ordering on B, $i \in P^B$ iff the i-th letter of w is 1.
(W.l.o.g. we restrict ourselves to nonempty words w to ensure that the B_W's are not empty.)
Define str:$\{0,1\}^* \rightarrow$powerset of $OStr(\{P\}^<)$ by $Str(w) := \{B \mid B \approx B_w\}$, (where ¨≈¨denotes the isomorphism-relation), and for $\mathbf{L} \subseteq \{0,1\}^*$ set $str(\mathbf{L}) := \cup\{str(w) \mid w \in \mathbf{L}\}$.

Then str(**L**) is a class of ordered {P}-structures.
In the following we consider complexity classes C, that contain L (=LOGSPACE). This ensures that the class of Turing machines witnessing that some language belongs to C is closed by adding subroutines that are in SPACE(clog(n)), c>0 a natural number.
Then we have for any formal language **L** ($\mathbf{L} \subseteq \{0,1\}^*$):

$$\mathbf{L} \in C \text{ iff } code(str(\mathbf{L})) \in C.$$

Hence, we lose nothing if we redefine a complexity class C as $\{K \in OStr(\tau^<) \mid \tau \text{ vocabulary}, code(K) \in C\}$.
So, we can look at complexity classes as classes of classes of ordered structures. As we have already indicated, this change enables us to apply methods of Logic and Model Theory to the investigation of complexity classes.
Various extensions of first order logic were developed in order to capture (describe) well-known complexity classes. The aim is to characterize the complexity classes by logics in the following sense:

Logic *L* characterizes complexity class C iff for all classes K of structures we have:
(*) $K \in C$ iff $K = Mod_L(\varphi)$ for some *L*-sentence φ. ($Mod_L(\varphi)$ denotes the class of all (finite) models of φ.)
We will be more precise below.

In all known cases where (*) holds, we have in addition:

(**) To each L-sentence φ one can effectively assign a Turing machine that accepts $Mod_L(\varphi)$ and is resource bounded according to C.

Assume that (*) and (**) hold. Then:

- we can consider the set of L-sentences as a universal programming language for C;
- if the set of L-sentences is recursive (recursive enumerable) we get a recursive (r.e.) representation of C;
- in many cases, every L-sentences is equivalent to an L-sentence describing the behaviour of a Turing machine that accepts $Mod_L(\varphi)$, i.e. we have a normal form theorem for L.

Without giving the definitions (see [EbFl] for more details) we quote some appropriate logics:

$\sum_1^1$ existential second order logic

FO(PFP) partial fixed-point logic

FO(IFP) inflationary (or inductive) fixed-point logic

FO(TC) transitive closure logic

FO(DTC) deterministic transitive closure logic.

All these logics are extensions of first order logic.

We have:

FO≤FO(DTC)≤FO(TC)≤FO(IFP)≤FO(PFP),

Where $L_1 \leq L_2$ means that every L_1-sentence is equivalent over finite structures to an L_2-sentence, hence L_2 is at least as expressive as L_1 (over finite structures). All inclusions are strict up to FO(IFP)≤FO(PFP). Here strictness is open since it is equivalent to P≠PSPACE (see below).

Now we would like to give more precise definitions in order to explain what it means to logically characterize a complexity class.

We say that

(a) L weakly captures C iff for all τ and all $K \subseteq OStr(\tau^{<})$:

K is $L(\tau^{<})$-axiomatizable $\Leftrightarrow$ K$\in$C.

(b) L strongly captures C iff for all τ and M$\subseteq$Str(τ):

M is $L(\tau)$-axiomatizable $\Leftrightarrow$ $M^{<}\in$C.

One can show that if L strongly captures C, then L weakly captures C.
Let us quote some important results:

Theorem:

(1) FO(DTC) weakly captures LOGSPACE

(2) FO(TC) weakly captures NLOGSPACE

(3) FO(IFP) weakly captures PTIME

(4) $\sum_1^1$ strongly captures NPTIME

(5) FO(PFP) weakly captures PSPACE

Point (4) of the theorem goes back to Fagin [Fag], as mentioned above, and can be seen as the first important result of Descriptive Complexity. (1) and (2) go back to Immerman (1987), (3) goes back to Immerman (1986) and Vardi (1982), and (5) to Abiteboul and Vianu (1989). Apart from (4), it's proved, that none of these results can be strengthened to strong capturing.

Since we can look at complexity classes as classes of languages as well as classes of ordered structures – as it was shown above – we get the following consequences of the theorem:

(a) PTIME=PSPACE iff FO(IFP)≡FO(PFP) on ordered structures

(b) PTIME=NPTIME iff FO(IFP)≡$\sum_1^1$ on ordered structures

(Point (b) can be strengthened by substituting ¨$\sum_1^1$¨ by ¨SO¨ (second order logic).)

One open problem concerns the question whether there is a logic which strongly captures the class PTIME. The great importance of such logic, if it exists, lies in the fact that it could be considered as a database language allowing expressing all feasible queries, and only these.

The question of the existence of a query language that could express exactly the polynomial time generic queries on a relational database was in fact posed by A. Chandra and D. Harel (1982), and since then much work in Finite Model theory has been devoted to tackle this problem.

Another open problem is the question whether the classes PTIME and NPTIME coincide. (Clearly we have $\mathrm{PTIME} \subseteq \mathrm{NPTIME}$.) Some researchers consider this question, known as the ¨P-NP-Problem¨ since around 1970, as the most important problem of Theoretical Computer Science. Obviously it is philosophically interesting question whether a problem which can be solved in polynomial time by a non-deterministic machine can also be solved in polynomial time by a deterministic machine, or, in other words, whether or not non-deterministic machines are stronger in this sense than deterministic ones.

Practical reasons for the importance of the P-NP-Problem are that there are a lot of significant problems in practice for which it is easy to see that they are in NP (for instance ¨the travelling salesman¨), but it is not known if there are deterministic polynomial algorithms (P-algorithms) to handle the problem. Furthermore, there was developed a very nice structure theory (S. Cook (1971) and R. Karp (1972)) showing that either all NP-problems, of which it is unknown whether they have a P-algorithm, have a P-algorithm (if $P=NP$) or none of these problems have a P-algorithm (if $P\neq NP$).

We have seen above, that by methods of Descriptive Complexity we are able to reduce the difficult P-NP-Problem to a logical one:

$$P=NP \text{ iff } FO(IFP)\equiv SO \text{ on ordered structures.}$$

The expressiveness of the quoted logic languages one can now investigate by purely model theoretical means. By such model theoretical methods (especially games) one can show for instance that

$$FO(DTC)<FO(TC)<FO(LFP).$$

This makes it clear that we can use (Finite) Model Theory as a powerful tool to handle questions concerning Theoretical Computer Science.

Without giving more details here, we would like to mention that Finite Model Theory has also important applications in the theory of (relational) databases, since a database is exactly a finite relational structure.

Model Theory has experienced a steady and strong development since the 1950's. It studies the connection between the syntax of logic languages and their semantics, that is, their models in a very general frame. Since the 1970's a lot of applications in ¨core mathematics¨ have been found.

The theorem of Loewenheim and Skolem (formulated in a general way by A. Tarski) and the compactness theorem represent the foundations of the discipline. The compactness theorem says that a set of sentences has a model if every finite subset has a model. This theorem has many applications and is perhaps the most used tool in Model Theory. The theorems of Loewenheim and Skolem guarantee that a set of sentences, which has an infinite model, has larger models in every infinite cardinality and has also some smaller models. Thus, it is impossible to find a set of sentences whose models have a given infinite cardinality.

The compactness theorem (and the completeness theorem) only holds in first order logic. That is why this logic is the most prominent language in Model Theory (first order logic is also sufficient to formalize virtually all important things in core mathematics, so we can build up mathematics by the axioms ZFC of set theory using first order logic). But there have also been studies on the model theoretical properties of extensions of first order logic, like infinitary logics and logics with new quantifiers.

Core notions of Model Theory are ¨elementary equivalence¨ and ¨elementary extension¨. Both notions have applications in model constructions, in algebra, set theory and other parts of mathematics. Another notion, which plays an important role in particular in some recent developments of the discipline, is the concept of a ¨type¨, which is a consistent set of formulas in the same free variables.

Complete axiomatizable theories are decidable (recursive). The question of decidability and completeness of theories and the development of sharp-witted methods to prove these properties were and still are an important motivation in mathematical logic. Some of these techniques, which have their origin in Model Theory, are:

Vaught´s test, Ehrenfeucht games (and other games), quantifier elimination, Robinson's method of model completeness, and others.

The study of ultraproducts also formed an important part of Model Theory in the past.

M. Morley gave a very significant impulse in the discipline in 1965. A theory T is called categorical in a cardinal $\kappa \geq \omega$, if T has only one model up to isomorphism in cardinality κ (that is all models of cardinality κ are isomorphic). Morley proved that a countable complete theory T that is categorical in some uncountable cardinal is categorical in all uncountable cardinals. Morley also introduced the notion of Morley-rank and totally transcendental theory which gave rise to a powerful dimension theory generalizing the notion of dimension in vector spaces and fields. Another important concept introduced by Morley was the ω-stability of a theory, which would play an important role in the future. Baldwin and Lachlan, after Morley, insisted on the study of uncountably categorical theories (also called ω_1-categorical theories). The Baldwin-Lachlan-Theorem (1971) says (among other things), that a countable theory T that is ω_1-categorical but not ω-categorical has ω-many countable models. The theorem gives also a certain structure theory of the countable models of T (using the notion of strongly minimal sets and dimension theory).

In this moment S. Shelah entered the scene and became the leading and most creative researcher of the area (and also in set theory) for the next decades. Shelah formulated a program to investigate the countable first order theories by classifying the models they have. For this it was necessary to see, in which case such a classification was possible. The main dichotomy developed by Shelah was the dividing of the theories in stable and unstable. A theory T is said to be stable if there does not exist a formula $\varphi(x,y)$ and tuples a_i, b_i $(i<\omega)$ such that $T|=\varphi(a_i,b_i)$ iff $i\leq j<\omega$. Shelah proved that a theory T that is not stable has 2^λ - many models in any cardinal $\lambda\geq\omega_1+T$ (note that for $\lambda\geq|T|$, 2^λ is the maximum number of models of cardinality λ which T can have).

This is called a ¨non-structural theorem¨, because in this case it is impossible to classify the models of T. On the other hand, if it is possible to classify the models of a theory T (and by the above result we can restrict ourselves to stable theories) then we say that T has a ¨structure theorem¨. Uncountable categorical theories have the ¨best¨ structure theorem, since they have only one model in every uncountable cardinality. (Moreover, they satisfy a strong form of stability, the ω-stability, this was an important fact on Morley's original work.)

The spectrum function of a complete first order theory is a map I(-,T) such that for any cardinal λ, $I(\lambda,T)$ is the number of models of T of cardinality λ. In the late 1960´s Morley conjectured that the spectrum function of a complete countable first order theory is non decreasing on uncountable cardinals, i.e. for all uncountable cardinals $\lambda<\kappa$, $I(\lambda,T)\leq I(\kappa,T)$. Shelah´s proof of the conjecture spanned almost 15 years and is the main topic of [Sh1]. Part of the proof is the development of the forking dependence relation on a stable theory. The theme through much of Shelah´s book is to find subsets of a model of a stable theory on which forking dependence is nice enough to admit a dimension theory.

The stability spectrum of T is the class of cardinals λ such that T is stable in λ (where ¨T is stable in λ¨ means: whenever M is a model of T of cardinality λ, the number of complete types over M is also λ; it holds that T is stable iff it is stable in some infinite cardinal). Shelah gave a description of the stability spectrum of T. He characterized the class of cardinalities $\lambda\geq 2^{|T|}$, such that T is stable in λ.

Under the assumption of stability, Shelah developed a sophisticated model theoretical machinery (forking, orthogonality, regular types, etc.) in order to classify the models of the theory. Perhaps the most important of these was (non-) forking, which serves as a notion of independence in models of stable theories. Meaning was given to the expression: ¨a is independent from b over A¨ (a, b tuples, and A some set in a model M of T).

Finite Model Theory evolves without much contact with the classical part of Model Theory. First motivations to develop a model theory of finite structures were B. Trakhtenbrot´s Theorem (1950) on the failure of the completeness theorem in the finite, and H. Scholz´s Spectral Problem (1952). Later Complexity Theory and Database Theory mainly motivated it.

A set of first order sentences which has for every $n<\omega$ a model of cardinality n, has an infinite model, by the compactness theorem. In the other case, if the hypothesis does not hold, there are only finitely many finite models and they can be axiomatized by one first order sentence. Hence, Finite Model Theory is a very special case of the general one, that is concerning also infinite models (as we have seen, the cardinality of infinite structures of a theory is not bounded (Loewenheim, Skolem)).

The methods of Finite and classical Model Theory are quite different. Transfinite or set theoretical combinatorics are replaced by finite combinatorics, classical results like the completeness theorem no longer hold when we restricted to finite models, and the set of sentences valid in all structures is no longer recursively enumerable by only admitting finite structures (Trakhtenbrot´s Theorem).

Many questions turn out to be trivial or irrelevant in the finite. On the other hand the spectrum problem, formulated by Scholz in 1952, which is trivial in the infinite (by Loewenheim, Skolem) represent a difficult and unsolved problem in the finite case:

The function assigning λ the value $I(\lambda,T)$ is called the spectrum function of the theory T, where I is, as above, the number of non-isomorphic models of T of cardinality λ. The spectrum problem is to find the set $S=\{\lambda| I(\lambda,T)>0\}$ for a given T. If $\lambda\geq\omega$ lies in S, then every infinite cardinal is in S (Loewenheim, Skolem). Hence, we can restrict the problem to the nontrivial case $S=\{n| I(n,\varphi)>0\}$ for n a natural number and φ a first order sentence. S is called the spectrum of φ. Open questions are whether the spectra are closed under complement (If S is a spectrum of φ, is $S'=\omega-S$ also a spectrum of some sentence ψ?) and the description of the spectrum of a sentence. It is interesting to notice that these problems are intimately linked to the P-NP-Problem.

The spectrum theorem says that a set $S\subseteq\omega$ is a spectrum of a first order sentence if and only if S (more exactly the binary code of S) is in NP.

Since in the infinite case the spectrum problem is trivial, it is natural to go ahead and ask for the values of the spectrum function in the infinite. In this way we arrive at the purposes of Shelah´s program formulated in his Classification Theory and briefly discussed in some aspects above, to compute the spectrum function for given T and infinite cardinals.

There is a feeling by many researchers that the techniques currently in use in Finite Model Theory seem to be too limited to tackle the open problems, for instance in Descriptive Complexity and Database Theory. In recent years much effort has been spent on trying to handle finite structures with the powerful tools of classical Model Theory, especially Stability Theory (Classification Theory), that is the sophisticated machinery developed by Shelah (and some others).

This area is called Embedded Finite Model Theory and studies finite models that are embedded in infinite structures, using classical techniques of Model Theory. Recently it has become clear that this way to look at finite structures represent a very promising application of Stability Theory and that the time is ripe for researchers in Descriptive Complexity to look back to some modern developments in the classical part of Model Theory.

There are two important trends in Embedded Finite Model Theory we would like to mention here. The first one is due to J. Baldwin, M. Benedikt and others and is motivated mainly by the theory of relational databases. Baldwin defines an embedded finite model as follows:

Let L be a language disjoint from S, and let M be an L-structure with domain U. For any (finite) S-structure A with domain contained in U, let M(A) denote the unique L∪S-structure that expands M and agrees with A on the interpretation of the predicates in S. Such an A is called an embedded finite model.

The second approach to deal with finite structures in the infinite context uses the notion of a smoothly approximable (or smoothly approximated) structure, which was introduced, it seems, by Lachlan.

Such an object is, by definition, a countable relational ω-categorical structure M that is the union of an increasing chain of finite homogeneous substructures of M. By a ¨homogeneous substructure of M we mean a subset A of M, such that for any finite tuples a and b chosen from A, a has the same type as b in M if and only if there is an automorphism of M which fixes A setwise and takes a to b. (A type of a tuple c is a consistent set of formulas in the same free variables satisfied by c, that is a set of formulas, such that every finite subset is consistent replacing the same free variables by the tuple c.)

One can show that ω-categorical, ω-stable structures are smoothly approximable (and smoothly approximable structures are simple, see the next chapters). Major progress in this area is done by Hrushovski and Cherlin [Hr], [ChHr].

2 Simple First Order Theories

2.1 From stability to simplicity

One of the reasons why researchers are interested in stable theories is that there is a notion of non-forking, introduced and developed essentially by Shelah [Sh1] which provides a concept of independence between sets in a structure to develop a dimension theory representing a generalization of well known concepts like ¨algebraic independence¨ in fields and ¨linear independence¨ in vector spaces. This is a very important motivation of investigation in Model Theory and Mathematics in general (first investigations about a general notion of independence in algebra were done, it seems, by van der Waerden in [vdW]). Forking/non-forking is a basic tool of Stability Theory for proving classification theorems and can be called a core notion.

The importance of simple theories is that they represent a generalization of stable theories and satisfy some nice properties, especially with respect to forking. Non-forking provides an independence notion for any simple structure, as well as for stable structures. In this way it is possible to study much better the structural properties of the models. The notion of independence made it possible to classify models. For instance, two vector spaces over the field of the rational numbers with the same dimension are isomorphic. Shelah developed stability theory in order to respond to the question of which theories (models of these theories) allow to define dimensions. (The theories which are not stable and - inside the stable theories - the theories which are not superstable, do not.) Shelah created a calculus and a theory of forking to treat in general the abstract independence notion. This calculus is valid in the general context of stable theories – and in the broader class of simple theories too.

The class of simple theories includes stable theories, but also many more such as the theory of the random graph. Moreover, many of the theories of particular algebraic structures, that have been studied recently (pseudofinite fields, smoothly approximable structures, a.o.), turn out to be simple. The interest is basically that a large amount of the machinery of stability theory, invented by Shelah, is valid in the broader class of simple theories.

Simple theories were introduced by Shelah in 1980 [Sh2] and remained unnoticed for some time. In the early 1990´s, E. Hrushovski noticed that the fact that the first order theory of an ultraproduct of finite fields is simple (and unstable) has far reaching consequences. His spectacular application to Diophantine Geometry attracted much attention to the general theory of simple theories. A. Pillay consequently prompted his Ph.D student B. Kim to study in the general context of simple theories a property that he and Hrushovki isolated and called the ¨Independence Property¨. Kim found a new characterization in terms of Morley sequences for the property ¨φ(x,y) divides over A¨. From this important characterization he managed to derive that for simple theories forking it is equivalent to dividing and forking satisfies the symmetry and transitivity properties [Kim1] (generalizing Shelah who proved this for stable theories). By these results one can define simplicity in terms of symmetry of the independence relation, what we are going to do in the following.
Since there are several examples of simple unstable theories that are of interest in their own right (mainly the random graph, pseudofinite fields, smoothly approximable structures), the study of simple theories turned out to be a main topic in Model Theory and many researchers are rewriting results in stable theories in the broader context of simple theories.

2. 2 Notation and some prerequisites

Let us first fix the notation.
T is a theory with no finite models, in a first order language L. We use x, y, … and a, b, c, … to denote (possible infinite) sequences of variables, elements.
A type p(x) in the tuple x of variables over a set A of parameters is a nonempty set of formulas with parameters in A, consistent with T. All formulas in p(x) have their free variables among the tuple x. An n-type is a type with n free variables.
Since a type p(x) is a consistent set of formulas (in the same free variables), it can be satisfied by a tuple c from some model of T. We say that c realizes the type p and write $c \models p(x)$.
If the type p is over the parameter set A, we say that the domain of p, dom(p), is A.
p, q, … denote types, α, β, κ, λ, μ, … ordinals or cardinals, card(A) the cardinality of a set A.
We work in a huge κ’-saturated model **C** of a given theory T, called the monster model or

universal domain, which will contain as elementary substructures all models of T we are interested in. **C** is also strongly κ'-homogeneous. (Recall that a model is κ-saturated if it realizes every 1-type over every set of cardinality <κ and a model is strongly κ-homogeneous, if whenever A, B are subsets of the model of cardinality <κ and f is a bijection between A and B which is an elementary map in the model, then f extends to an automorphism of the model.) The existence of such a monster model is guaranteed by the following fact [Ba1],[ChK]:

Fact :
If κ' is a regular cardinal > card(L), then there is a model **C** of T which is κ'-saturated and strongly κ'-homogeneous.

(Now choose κ' larger than all the models of T we are interested in.)

Sets A, B, C, ... are always subsets of **C**. It is assumed that the cardinalities of the sets and (elementary) submodels are strictly less than κ'. We say that a set is bounded, if its cardinality is strictly less than κ', and the set is unbounded otherwise.
We often write AB, Ab to denote the unions A∪B, A∪{b}, where A, B are sets, b a tuple or a (infinite) sequence.
For a set A let T(A) denote the theory of a model (**M**,A) in the language L(A) obtained from model **M** of T by adding constants c∈A. An A-automorphism of **M** is an automorphism of **M** fixing A pointwise. Aut(**M**/A) is the group of all A-automorphisms of **M**.

tp(c/B) denotes the complete type realized by c over B, where complete means: for all formulas φ(x) over B hold φ(x)∈tp(c/B) or ¬φ(x)∈tp(c/B) (where c, x are tuples of the same length.).
We shall extend the notion of a type in an obvious way working with infinite sequences instead of finite tuples. Let I be an arbitrary (infinite) index set. We extend the formal language introducing new variables $\{x_i : i\in I\}$. The free variables of formulas of the obtained language are in $\{x_i : i\in I\}$. Then an I-type p is a consistent set of formulas with free variables in $\{x_i : i\in I\}$. It is complete, if for any formula φ of the language it holds that, either φ∈p or

$\neg\varphi\in p$. $S_I(A)$ denotes the set of complete I-types with parameters in A. Now we can consider also types of infinite sets or sequences over some parameter set. If B is a set, enumerated as a sequence of cardinality $\leq$card(I), then tp(B/A) is a complete I-type consisting of formulas $\varphi(x,a)$, where $a\subseteq A$ and x is a finite subtuple of $(x_i : i\in I)$.

The following fact is highly important and in the following we shall make tacitly use of it. [Bue1],[ChK]

Fact :

Let a, b be sequences from the monster model **C** (of cardinality less than κ'). Then the following holds:

tp(a/A)=tp(b/A)   if and only if   there is an A-automorphism of **C** sending a to b.

This fact is guaranteed by the strongly κ'-homogeneity of the monster model **C**, which means, that every partial automorphism (elementary map) between two subsets of cardinality less than κ' can be extended to an automorphism of **C**.

Sequences, or sets C, D, with tp(C/A)=tp(D/A), we call A-conjugated.

If $\varphi(x)$ is a formula, where $x=(x_1,\ldots,x_k)$, k>0, then $\varphi(\mathbf{M})$ will denote the set $\{m\in\mathbf{M} : \mathbf{M}\models\varphi(m)\}$, where $m=(m_1,\ldots,m_k)$.

Defnition 2.2.1:

Let **M** be a L-structure. A subset X of $\mathbf{M}^k$ is

1. *definable* if there are a tuple $c\subset\mathbf{M}$ of parameters and an L(c)-formula $\varphi(x_1,\ldots,x_k,c)$ such that $X=\varphi(\mathbf{M})$, and
2. *type-definable*, if X is the intersection of an arbitrary family of definable subsets. That is, X is the set of realizations of some partial type.

If all the parameters used for some (type-)definable set X are contained in a subset A of **M**, we say that X is (type-) definable *over* A, or A-(type-) definable. We shall often identify a formula with the set it defines (where the model is given implicitly).

The proof of the following fact uses a topological argument and can be found in any textbook of model theory.

Fact : Let $D\subseteq\mathbf{C}^k$ be type-definable (over some set of parameters). If D is A-invariant, that is, if for any $f\in Aut(\mathbf{C}/A)$, $f(D)=D$, then D is type-definable over A.
If D is definable and A-invariant, then D is definable over A.
If $p(x,a_0)$ is a type over Aa_0 and $tp(a_0/A)=tp(a_1/A)$, then $p(x,a_1)$ is an image of $p(x,a_0)$ under an A-automorphism.
S(A) is the set of all complete types over A and $S_n(A)$ the set of all complete n-types over A, that is, the complete types in an n-tuple of variables. Then $S(A)=\cup\{S_n(a)|\ n<\omega\}$.

Definition 2.2.2 : Let A be a set of parameters, $p\in S(A)$. The sequence $(a_i : i\in I)$ is a sequence *of type p* (or *in type p*), if $a_i \models p$ for all $i\in I$.
If the index set I is ordered, the sequence $(a_i : i\in I)$ is *n-indiscernible over A*, if for all $i_1<i_2<\ldots<I_n$ the type $tp(a_{i1}\ldots a_{in}/A)$ does not depend on the choice of indices. It is *indiscernible over A*, if it is n-indiscernible over A for all $n<\omega$.

Let us quote two very useful combinatorial principles:

Theorem 2.2.3 Ramsey's Theorem : Suppose X is an infinite set and the set of unordered n-tuples of X is painted in k different colors. Then there is an infinite monochromatic subset Y, i.e. a subset $Y\subseteq X$ whose n-tuples all have the same color.

Theorem 2.2.4 Erdös-Rado Theorem : Suppose κ is a cardinal, X is an infinite set of cardinality $(2^\kappa)^+$, and the set of unordered n-tuples of X is painted in κ different colors. Then there is an infinite monochromatic subset $Y\subseteq X$ of size κ^+.

Sometimes one finds another formulation of these two theorems: The notation

$$\kappa\to(\lambda)_\nu^\mu$$

means that if A and B are sets of cardinality κ and ν respectively, and $f:[A]^{\mu}\rightarrow B$ is a function from the set of unordered μ-tuples of A to the set B, then there is a subset $A'\subseteq A$ of cardinality λ, such that f is constant in A', i.e. there is some $b\in B$ such that for all $a\in[A]^{\mu}$, $f(a)=b$.
Then Ramsey's Theorem says that for all n, $m<\omega$, $\omega\rightarrow(\omega)_m{}^n$. The Erdös-Rado Theorem says that for all cardinals κ and all $n<\omega$, $(2^{\kappa})^+\rightarrow(\kappa^+)_{\kappa}{}^n$.
A stronger form of the Erdös-Rado Theorem says that for all cardinals κ and all $n<\omega$, $(\Im_n(\kappa))^+\rightarrow(\kappa^+)_{\kappa}{}^{n+1}$, where $\Im_n(\kappa)$ is the beth-function.

The following result about indiscernibles is often useful. Its proof (see e.g. [Kim1]) uses the Erdös-Rado Theorem, a technique discovered by Morley to prove his Omitting Types Theorem ([ChK, Theorem 7.2.2]) and further set theoretical combinatorics.

Proposition 2.2.5 : Let $\kappa\geq card(T)$ a cardinal and let $\lambda=\Im_{\mu}$, where $\mu=(2^{\kappa})^+$. Let A be a set of cardinality $\leq\kappa$ and let $(a_i : i<\lambda)$ be a sequence of sequences a_i of length $\leq\kappa$. Then there is an A-indiscernible sequence $(b_i : i<\omega)$ such that for every $n<\omega$ there exist $i_0<\ldots<i_n<\lambda$ with $tp(b_0,\ldots,b_n/A)=tp(a_{i0},\ldots,a_{in}/A)$.

Remark :
This Proposition implies the following result, which we will need sometimes:
If I is a B-indiscernible sequence and $B\subseteq C$, then there exists another sequence I' which is B-isomorphic to I (that is, there is an automorphism fixing B and mapping I to I') and C-indiscernible. Hence, there is also a set C', an B-isomorphic image of C, such that I is C'-indiscernible.

Proof : By compactness, we may assume that I is the sequence $(a_i : i<\lambda)$ of cardinality λ, like in the Proposition. Then, by the Proposition, there is some C-indiscernible sequence $(b_i : i<\omega)$ with the properties given in the Proposition (substituting A by C). Consider the set X of formulas expressing that $I'=(x_i : i<\lambda)$ is a C-indiscernible sequence, B-isomorphic to I. Any finite subset of X in n variables is satisfiable by some $(a_{i0},\ldots,a_{in})\subseteq I$, $i_0<\ldots<i_n<\lambda$, like in the

Proposition (by C-indiscernibility of $(b_i : i<\omega)$ and B-indiscernibility of I). By compactness, X is consistent.

Now, mapping I' by a B-automorphism to I, we obtain the B-automorphic image C' of C with the property that I is C'-indiscernible.

q.e.d.

We shall now introduce Shelah's eq-construction, which allows us to deal with equivalence classes (modulo a definable equivalence relation) just as if they were real elements (or tuples) in the structure.

For L a language and T a theory in L, L^{eq} and T^{eq} are defined as follows (we suppose that L and T are 1-sorted). Let E be the set of all formulas E(x,y) over $\emptyset$ such that for some n and every model **M** of T, E defines an equivalence relation on $\mathbf{M}^n$. Let $I=\{I_E : E\in E\}$ be a collection of distinct sorts. For each $E\in E$ let f_E be a function symbol taking n-tuples from the sort $i_=$ into the sort I_E. Finally let L^{eq} be the I-sorted language which contains $\{f_E : E\in E\}$ and for each element of L a corresponding element whose arguments are required to range over the sort $i_=$. The axioms for T^{eq} are the axioms for T restricted to the sort $i_=$, together with all statements expressing: f_E is a surjective map of n-tuples from $i_=$ onto i_E such that $\forall xy(E(x,y)\leftrightarrow f_E(x)=f_E(y))$, where x, y are n-tuples. From hereon we will identify T with its copy on $i_=$ in T^{eq}.

Statements made in T^{eq} can always be reduced to statements in T. This is made precise in the following lemma, which is easily proved by induction on formulas.

Lemma 2.2.6 : For any formula $\varphi(v_0,\ldots,v_n)$ of L^{eq}, with v_j a variable of sort I_{Ej}, there is a formula $\varphi^*(w_0,\ldots,w_n)$ of L such that

$$T^{eq}\models\forall w_0\ldots w_n(\varphi(f_{E0}(w_0),\ldots,f_{En}(w_n))\leftrightarrow\varphi^*(w_0,\ldots,w_n)).$$

Let T be a complete theory in L with monster model **C**. Let $\mathbf{C}^{eq}$ be an expansion of **C** to a model of T^{eq}. (For E a formula defining an equivalence relation on n-tuples let $(\mathbf{C}^{eq})_{iE}=\mathbf{C}^n/E$ the E-equivalence classes on $\mathbf{C}^n$, and f_E be the quotient map.) Notice that $\mathbf{C}^{eq}$ is obtained from **C** simply by closing under the functions of the language L^{eq}. This observation makes it clear that $\mathbf{C}^{eq}$ is the unique model N of T^{eq} with $\mathbf{C}=N_{i=}$. Furthermore, an automorphism f of **C** can be extended uniquely to an automorphism of $\mathbf{C}^{eq}$. It is easy to sea that $\mathbf{C}^{eq}$ is a monster model of T^{eq}.

Definition 2.2.7 : Let T be a complete theory, possibly many sorted, with monster model **C**.

(i) If D is a definable set in $\mathbf{C}^n$, d is called the *canonical parameter* for D if f(D)=D $\leftrightarrow$ f(d)=d for all f$\in$Aut(**C**). (This means, that f fixes D setwise if and only if f fixes d pointwise.)

(ii) If every definable set has a canonical parameter in **C**, we say that T has *elemination of imaginaries* or T has *built-in-imaginary elements*.

Proposition 2.2.8 : Given a complete theory T, T^{eq} has elimination of imaginaries.

Proof. Let **C** be the monster model of T and D=φ(**C**,a), where φ(x,y) is a formula of L, a, x, y are tuples. Let E(y,y') be the equivalence relation: E(y,y') iff $\forall x(\varphi(x,y)\leftrightarrow\varphi(x,y'))$. Then, for all b and c, $\models$E(a,b) iff φ(**C**,a)=φ(**C**,b). Hence, an automorphism of $\mathbf{C}^{eq}$ permutes the set D if and only if it fixes a/E. Thus, a/E is a canonical parameter for D in $\mathbf{C}^{eq}$. In a similary way one can show that if D is a definable subset of $(\mathbf{C}^{eq})^n$, for some n, then there is also a canonical parameter for D in $\mathbf{C}^{eq}$.
q.e.d.

From hereon, unless stated otherwise, we restrict our attention to theories with elimination of imaginaries and we will work in $\mathbf{C}^{eq}$, however, we usually will omit the notation ^{eq}.

Definition 2.2.9 : Let A be a set of parameters.

(i) An element a is *definable over A* if it is fixed under all A-automorphisms.

(ii) An element a is *algebraic over A* if it has only finitely many images under A-automorphisms.

(iii) A set X is *A-invariant* if it is stabilized setwise under all A-automorphisms.

(iv) Two sets X and Y are *A-conjugate* if there is an A-automorphism mapping X to Y.

The *algebraic closure* of A, denoted by acl(A), is the set of all elements algebraic over A. The *definable closure* dcl(A) over A is the set of all elements definable over A. If we want to emphasize that we take the algebraic or definable closure in $\mathbf{C}^{eq}$, we denote this by $acl^{eq}(A)$ and $dcl^{eq}(A)$.

Lemma 2.2.10 : Let A and a be as above. Then a is definable over A if and only if there is an L(A)-formula $\varphi(x)$ whose sole realization is a. It is algebraic over A if and only if there is an L(A)-formula realized by a and having only finitely many realizations.

Proof. $\Leftarrow$ is obvious in both cases, as $\varphi(\mathbf{C})$ is invariant under Aut($\mathbf{C}$/A). For the other direction, suppose $a\in dcl(A)$ and consider p(x)=tp(a/A). By κ-saturation of $\mathbf{C}$, it must be inconsistent to say $p(x)\cup p(x')\cup\{x\neq x'\}$. But this means that $q(x)\cup q(x')\cup\{x\neq x\}$ is inconsistent for some finite part q of p, and $\varphi(x)=\wedge q(x)$ will do. The case $a\in acl(A)$ is similar.
q.e.d.

Now it is easy to see that dcl(), acl() are idempotent.

2.3 Dividing and forking

Definition 2.3.1 : Let $k<\omega$. A formula $\varphi(x,a)$ *k-divides over A* if there is a sequence $(a_i : i<\omega)$ of type tp(a/A) such that $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent, i.e. any finite subset of size k is inconsistent.
A partial type p(x) k-divides over A if there is a formula $\varphi(x)$ implied by p(x) which k-divides over A. A formula or a partial type divide over A if they k-divide for some $k<\omega$.
A partial type p(x) forks over A if there are $n<\omega$ and formulas $\varphi_0(x),\ldots,\varphi_n(x)$ such that p(x) implies $\vee_{i<n}\varphi_i(x)$, and each $\varphi_i(x)$ divides over A.

If $A \subseteq B$ and $p(x) \in S_n(B)$ (or $p \in S_I(B)$, for an I-type) does not fork over A, we call p a *non-forking extension* of $p \lceil A$. ($p \lceil A$ denotes the partial type contained in p whose parameters are restricted to the set A.)

Remark : Let $k<\omega$. For every formula $\varphi(x,y)$ and every set A there is a partial type q(y) over A such that for any a holds:

$\varphi(x,a)$ k-divides over A if and only if $\models q(a)$.

The type q(y) will express that there is a sequence $(y_i : i<\omega)$ of type tp(y/A) such that the set $\{\varphi(x,y_i) : i<\omega\}$ is k-inconsistent.

Some basic properties of dividing and forking are expressed in the following

Lemma 2.3.2 :

1. Dividing implies forking.
2. If p and q are two partial types which fork over A, so does $p \vee q$.
3. If $p \vdash q$ and q divides (forks) over A, so does p. (monotonocity)
4. φ k-divides over A if and only if it k-divides over all finite tuples $a \in A$.
5. In the definition of dividing, we may require the sequence $(a_i : i<\omega)$ to be indiscernible over A.
6. A partial type p(x) k-divides (forks) over A if and only if there is a finite conjunction $\varphi(x)$ of formulas in p which k-divides (forks) over A.
7. No $p \in S_n(A)$ divides over A. (existence of nondividing extensions)
8. Let $A \subseteq B \subseteq C$. If tp(a/C) does not divide (fork) over A, then it does not divide (fork) over B, and tp(a/B) does not divide (fork) over A. (partial transitivity)
9. tp(a/Aa) divides (forks) over A if and only if $a \notin acl(A)$.
10. Let $A \subseteq B$. tp(a/B) does not divide (fork) over A if and only if for each finite tuple $b \in B$ tp(a/b) does not divide (fork) over A. (finite character)

Proof: 1.-3. and 8. follow quickly from the definition of dividing and forking.

4.: The direction from left to right is clear. For the other direction, suppose that $\varphi(x,c)$ k-divides over all finite $a \in A$ and consider the set of formulas expressing that there is a sequence $(c_i : i<\omega)$ of type $tp(c/A)$, such that $\{\varphi(x,c_i) : i<\omega\}$ is k-inconsistent. By hypothesis and compactness this set is consistent, hence $\varphi(x,c)$ k-divides over A.

5.: Let $n<\omega$. Consider the set X of formulas in the variables $(x_i : i<\omega)$ saying that all the x_i are distinct, of type $tp(a_0/A)$ and that $(x_i : i<\omega)$ is an n-indiscernible sequence. In order to show that this set is consistent, consider the finite subset X_0 of X:

$X_0=\{\varphi_m(x_{i1},\ldots,x_{in})\leftrightarrow\varphi_m(x_{j1},\ldots,x_{jn}) : x_{ik}, x_{jk} \in (x_i : i<\omega) \text{ for } 0<k<n+1, \varphi_m\in L, m<r\}$ where the indices are increasing and $r<\omega$ is fixed. So we can color the n-tuples of $(a_i : i<\omega)$ in 2^r different colors, according to which of the r formulas $\varphi_m(x_{i1},\ldots,x_{in})$, $m<r$, they satisfy. By Ramsey's Theorem there is an infinte monochromatic subset of $(a_i : i<\omega)$ satisfying X_0. By compactness X is consistent. Since $n<\omega$ was arbitrary, we can suppose that there is an indiscernible sequence of $type(a_0/A)$.

6: Consider a formula $\varphi(x,a)$ implied by p which k-divides over A, as witnessed by a sequence $(a_i : i<\omega)$. Then there is a finite part $q(x,b)$ such that $q(x,b)\vdash\varphi(x,a)$; since $tp(a_i/A)=tp(a/A)$, there is a sequence $(b_i : i< \omega)$ with $tp(b_i/A)=tp(b/A)$ and $q(x,b_i)\vdash\varphi(x,a_i)$. Then this sequence witnesses that $\wedge q$ k-divides over A; the case of forking is obvious.

Clearly 6. implies 7.

9: From left to right it is sufficient to show that any infinite indiscernible sequence over A must be indiscernible over acl(A) too: So let I be indiscernible over A. By Ramsey's Theorem and compactness we can find an sequence $J\subseteq I$ indiscernible over acl(A) having the same type over A like I. Hence there is an automorphism $f\in Aut(\mathbf{C}/A)$ sending J to I and fixing A. Then I must have been indiscernible over f(acl(A))=acl(A).

For the other direction of 9. consider the formula x=a.

10: Follows from 6.

q.e.d.

Remark : From hereon, by 2.3.2.5, we will normally work with indiscernible sequences in the definition of dividing.

Intuitively, dividing is the right notion of dependence. Suppose that tp(c/Ab) divides over A. tp(c/Ab) is a (dividing) extension of the type tp(c/A), and, clearly, every realization of tp(c/Ab) realizes tp(c/A). By 2.3.2.6 there is a $\varphi(x,b)\in tp(c/Ab)$ dividing over A. Hence, there is a sequence $(b_i| \, i<\omega)$ with $tp(b/A)=tp(b_i/A)$, $i<\omega$, generating ω-many A-isomorphic images of tp(c/Ab), taking b to b_i, $i<\omega$. Thus, the set of realizations X of tp(c/Ab) breaks up that of tp(c/A) into ω-many pieces X_i, each X_i is an A-automorphic image of X. This means in some sense that c satisfies more relations with Ab than it does with A.

Why, then is forking introduced over dividing? With forking, we have the following extension axiom (for types).

Lemma 2.3.3 : Let $A\subseteq B$ and let p be a type over B which does not fork over A. Then there is a complete type q over B extending p, which does not fork over A.

Proof: We show that for any B-formula $\varphi(x,b)$ either $p\cup\{\varphi\}$ or $p\cup\{\neg\varphi\}$ does not fork over A. So suppose otherwise. Then there are formulas $\varphi_1(x),\ldots,\varphi_m(x)$ and $\psi_1(x),\ldots,\psi_n(x)$, all of them dividing over A, such that $p\cup\{\varphi\}\vdash\vee_i\varphi_i$ and $p\cup\{\neg\varphi\}\vdash\vee_j\psi_j$. Hence $p\vdash(\vee_i\varphi_i)\vee(\vee_j\psi_j)$, so p forks over A, a contradiction.

As non-forking over A is a local property by Lemma 2.3.2.6, it is closed under unions of chains. (That is, if $p=p_0\subseteq p_1\subseteq p_2\subseteq\ldots\subseteq p_n\subseteq\ldots$ is a chain of non-forking extensions of p, then the union of this chain is a non-forking extension too. Otherwise, by 2.3.2.6 (or by compactness), there would be an i such that p_i forks over A.) The existence of a non-forking completion of p now follows from Zorn's Lemma.

q.e.d.

Remark : An alternative proof of Lemma 2.3.4 is to show that the set

$$p(x)\cup\{\varphi(x) : \neg\varphi(x)\in L(B) \text{ divides over } A\}$$

is consistent (it is clear that this set is also complete). Suppose not, then there is a finite subset which is inconsistent. Hence $p\vdash\neg\varphi_i$ for some $0\leq i\leq n$, and $\neg\varphi_i$ divides over A. It follows that p forks over A, a contradiction.

The following definition gives particular examples of non-forking extensions.

Definition 2.3.4 : Let $A \subseteq B$ and p be a partial type over B. We say that p is *finitely satisfiable* in A if every finite conjunction of formulas in p is satisfied by some tuple in A.
If **M** is a model of T, $p \in S(\mathbf{M})$ and $\mathbf{M} \subseteq B$, then an extension $q \in S(B)$ of p is called a *coheir* of p if it is finitely satisfiable in **M**.

Lemma 2.3.5 : Let $A \subseteq B$ and q be a partial type over B which is finitely satisfiable in A. Then q has a completion $p \in S(B)$ which is finitely satisfiable in A. If b, $b' \in B$ with $tp(b/A)=tp(b'/A)$ and $c \models p$, then $tp(cb/A)=tp(cb'/A)$. Furthermore, p does not fork over A.

Proof. As finite satisfiability is closed under unions of chains, it is enough to show that for every B-formula φ either $q \cup \{\varphi\}$ or $q \cup \{\neg\varphi\}$ is finitely satisfiable in A. So suppose not. Then there are finite bits $q_0 \subseteq q$ and $q_1 \subseteq q$ such that $q_0 \cup \{\varphi\}$ and $q_1 \cup \{\neg\varphi\}$ are both not satisfied by any tuple in A. But then $q_0 \cup q_1$ is not satisfied by any tuple in A, a contradiction. So q can be completed to a type which is finitely satisfiable in A.
Now suppose b, $b' \in B$ with $tp(b/A)=tp(b'/A)$ and $c \models p$. If there is a formula φ with $\models \varphi(c,b) \wedge \neg\varphi(c,b')$, then by finite satisfiability there is $a \in A$ with $\models \varphi(a,b) \wedge \neg\varphi(a,b')$, contradicting $tp(b/A)=tp(b'/A)$.
Finitely, let q be finitely satisfiable in A, and suppose $q(x) \vdash \vee_{i<n}\varphi(x,b_i)$. By finite satisfiability, there must be some $a \in A$ and some $i<n$ with $\models \varphi_i(a,b_i)$. But then for any A-indiscernible sequence I with $b_i \in I$ we have $\models \varphi_i(a,b')$ for all $b' \in I$., so $\varphi_i(x,b_i)$ cannot divide over A.
q.e.d.

Remark :
If p is a type over a model **M**, then p is finitely satisfiable in **M** by consistency of the type p. If $B \supseteq \mathbf{M}$, then p can be viewed as a partial type over B. So by the first part of Lemma 2.3.5, p has a completion q over B, which is a coheir of p. Therefore types over models always have coheirs. However, not every type over B which does not divide (or fork) over $\mathbf{M} \subseteq B$ is a coheir of its restriction to **M**.

Proposition 2.3.6 : The following are equivalent:

1. tp(a/Ab) does not divide over A.
2. For any A-indiscernible sequence I with $b\in I$, there is a tuple a' realising tp(a/Ab) such that I is indiscernible over Aa'.
3. If I is an A-indiscernible sequence with $b\in I$, then there is an Ab-automorphic image J of I which is indiscernible over Aa.

Proof. The equivalence of 2. and 3. follows by taking an Ab-automorphism mapping a to a' and J to I.

Suppose tp(a/Ab) does not divide over A, and let I be an A-indiscernible sequence with $b\in I$.

Write tp(a/Ab)=p(x,b).

Claim: $q(x):=\cup_{b'\in I}p(x,b')$ is consistent.

If not, then for some formula $\varphi(x,b)\in p(x,b)$ the conjunction $\wedge_{b'\in I}\varphi(x,b')$ is inconsistent, and hence k-inconsistent by compactness and indiscernibility of I, for some $k<\omega$. So p(x,b) divides over A, a contradiction. This proves the claim.

Let $\Gamma(x)$ be the set of formulas expressing that I is an indiscernible sequence over Ax, and let $\Gamma_0(x)$ be a finite subset. A formula in $\Gamma_0(x)$ is of the form: $\varphi(b_{i1},\ldots,b_{im},x,a)\leftrightarrow\varphi(b_{j1},\ldots,b_{jm},x,a)$, where a is a tuple of elements of A and b_{ik}, $b_{jk}\in I$. If $\Gamma_0(x)$ contains n formulas of this form, then for an arbitrary interpretation c of x there are 2^n possible combinations for any m-tuple from I to satisfy the n subformulas on the left side of the equivalences by substituting the $b_{i1},\ldots,b_{im}$. Now let us paint the increasing m-tuples of I in 2^n different colors, according to the way in which they satisfy the subformulas on the left side in $\Gamma_0(x)$. So we can find – by Ramsey's Theorem - an infinite subsequence J of I such that all increasing m-tuples of J are of the same color, whence satisfy the set $\Gamma_0(c)$ (the elements of I in Γ_0 are substituted by the elements of J). Since J and I have the same type over A (by indiscernibility), there is a corresponding realization c' of $\Gamma_0(x)$, mapping J by an A-automorphism to I. When we choose for c a realization of q(x), then c' realizes $q(x)\cup\Gamma_0(x)$. (Note that c' then automatically realizes p(x,b)=tp(a/Ab), since $b\in I$.)

By compactness, $q(x)\cup\Gamma(x)$ is consistent. Now we take for a' a realization of $q(x)\cup\Gamma(x)$.

For the converse, suppose that tp(a/Ab) divides over A. Then there is a infinite A-indiscernible sequence I with $b\in A$, and a formula $\varphi\in tp(a/Ab)$, such that $X=\{\varphi(x,b') : b'\in I\}$

is inconsistent. By 2. there is a tuple a’ realizing tp(a/Ab) such that I is Aa’-indiscernible. So a’ realizes X, a contradiction. Hence tp(a/Ab) does not divide over A.
Q.e.d.

Proposition 2.3.7 : Let $A \subseteq B$, and suppose $tp(a_i/Ba_0,\ldots,a_{i-1})$ does not divide over $Aa_0,\ldots,a_{i-1}$ for all $i \leq n$. Then $tp(a_0,\ldots,a_n/B)$ does not divide over A.

Proof. It is sufficient to show that $tp(a_0,\ldots,a_n/Ab)$ does not divide over A for every tuple $b \in B$. Let I be an A-indiscernible sequence with $b \in I$. By hypothesis, $tp(a_0/Ab)$ does not divide over A. Now suppose that we have found $a'_0,\ldots,a'_{i-1}$ realising $tp(a_0,\ldots,a_{i-1}/A)$ such that I is indiscernible over $Aa'_0\ldots a'_{i-1}$. If a’ is such that $tp(a_0\ldots a_{i-1}a_i/Ab)=tp(a'_0\ldots a'_{i-1},a'/Ab)$, then $tp(a'/Aba'_0\ldots a'_{i-1})$ does not divide over A by invariance of dividing under automorphisms. By Proposition 2.3.6 there is a'_i realizing $tp(a'/Aba'_0\ldots a'_{i-1})$ such that I is indiscernible over $Aa'_0\ldots a'_i$. Inductively, we find $a'_0\ldots a'_n \models tp(a_0\ldots a_n/Ab)$ such that I is indiscernible over $Aa'_0\ldots a'_n$, and we finish by Proposition 2.3.6.
q.e.d.

Note that in the proof above, the tuples a_i may be infinite.

2.4 Simplicity

Definition 2.4.1 : A set A is *independent* of C over B, denoted $A \perp_B C$, if tp(a/BC) does not divide over B for any finite tuple $a \in A$. To express the negation we write $A \not\perp_B C$.
A (complete first-order) theory is *simple* if independence is a symmetric notion (that is $A \perp_B C$ if and only if $C \perp_B A$, for all subsets A, B, C of the monster model).
We shall call a structure *simple* if its theory is.

Definition 2.4.2 : Let $\varphi(x,y)$ be a formula and $k<\omega$. The rank $D(.,\varphi,k)$ is defined inductively on partial types as follows:

1. $D(p(x),\varphi(x,y),k) \geq 0$ if p(x) is consistent.

2. $D(p(x),\varphi(x,y),k)\geq n+1$ if there is a tuple b such that $D(p(x)\wedge\varphi(x,b),\varphi,k)\geq n$, and $\varphi(x,b)$ k-divides over the domain of p. ($p(x)\wedge\varphi(x)$ means $p(x)\cup\{\varphi(x)\}$.)
3. $D(p(x),\varphi(x,y),k)=n$ if $D(p(x),\varphi(x,y),k)\geq n$ and not $D(p(x),\varphi(x,y),k)\geq n+1$.
4. $D(p(x),\varphi(x,y),k)=\infty$ if $D(p(x),\varphi(x,y),k)\geq n$ for all $n<\omega$.

Remark 2.4.3 :

(i) Let p be a partial type over some parameters A. Then $D(p(x,A),\varphi,k)\geq n$ can be expressed by a partial type over A. (This type says that there is a sequence $(b_j : 0\leq j<n)$ of tuples such that $\{p(x,A)\cup\varphi(x,b_j) : j<n\}$ is consistent and $\varphi(x,b_j)$ k-divides over $dom(p)\cup\{b_i : i<j\}$, for every $j<n$.)

(ii) It is clear by the definition, that $D(p(x),\varphi(x,y),k)\leq D(q(x),\varphi(x,y),l)$ if $p\vdash q$ and $k\leq l$, since $D(p(x),\varphi(x,y),k)\geq n$ implies $D(q(x),\varphi(x,y),l)\geq n$, for all $n<\omega$.

We shall write $D(a/A,\varphi,k)$ for $D(tp(a/A),\varphi,k)$.

Definition 2.4.4 : Let $\varphi(x,y)$ be an L-formula, $k<\omega$, and α an ordinal. A *(φ,k)-tree of heigth α* is a sequence $(a_\mu : \mu\in\omega^{<\alpha})$ such that

(i) for all $\mu\in\omega^{<\alpha-}$ the set $\{\varphi(x,a_{\mu\wedge i}) : i<\omega\}$ is k-inconsistent (where $\alpha-$ is the least ordinal whose successor is $\geq\alpha$ and $\mu\wedge i$ is the concatenation of the sequence $\mu=(\mu_j : j<\beta)$ with i, for some $\beta<\alpha-$)

(ii) for every $\mu\in\omega^{\alpha}$, $\{\varphi(x,a_{\mu\restriction i}) : i<\alpha\}$ is consistent (where $\mu\restriction i$ denotes the sequence $(\mu_j : j<i)$).

We say that φ has the *k-tree property* if there is a (φ,k)-tree of heigth ω. Finally, φ has the *tree property* if φ has the k-tree property for some $k<\omega$.
Furthermore, we say that the theory T has the tree property if there is a formula which has the tree property.

Remark 2.4.5 :

It is clear by compactness, that if φ has the k-tree property, then for every cardinal κ and every cardinal λ there exist parameters $(a_s : s\in\lambda^{<\kappa})$ such that for any $s\in\lambda^{<\kappa}$ the set $\{\varphi(x,a_{s\wedge j}) : j<\lambda\}$ is k-inconsistent, and for any $\alpha\in\lambda^{\kappa}$ the set $\{\varphi(x,a_{\alpha\lceil j}) : j<\kappa\}$ is consistent.

Definition 2.4.6 : Let α be an ordinal, $\varphi(x,y)$ a formula. A dividing chain of length α in $\varphi(x,y)$ is a sequence $(a_i : i<\alpha)$ such that $\{\varphi(x,a_i) : i<\alpha\}$ is consistent, and for all $i<\alpha$ holds: $\varphi(x,a_i)$ k_i-divides over $\{a_j : j<i\}$, for some $k_i<\omega$. It is a k-dividing chain if $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all $i<\alpha$. We say that φ (k-)divides α times if there is a (k-)dividing chain of length α in φ.

If p(x) is some partial type over A, and $p\cup\{\varphi(x,a_i) : i<\alpha\}$ is consistent, and $\varphi(x,a_i)$ divides over $A\cup\{a_j : j<i\}$ for all $i<\alpha$, we say that $(a_i : i<\alpha)$ is a dividing chain in φ, consistent with p.

Proposition 2.4.7 : The following are equivalent:

1. There is a (φ,k)-tree of heigth n, for all $n<\omega$.
2. There is a (φ,k)-tree of heigth α, for all ordinals α.
3. φ k-divides n times, for all $n<\omega$.
4. φ k-divides α times, for all ordinals α.
5. For every $n<\omega$, there is some type p with $D(p,\varphi,k)\geq n$.
6. For every ordinal α, there is some type p with $D(p,\varphi,k)\geq\alpha$.

Proof. 1.⇒2. and 3.⇒4. are true by compactness, 6.⇒5. is trivial.

2.⇒3.: Let the sequence $(a_\mu : \mu\in\omega^{<n})$ be a (φ,k)-tree of height n. First, by compactness, we can extend the sequences $(a_{\mu\wedge i} : i<\omega)$, for $\mu\in\omega^{<n-1}$, as long as we want (for any infinite ordinal λ, we obtain a k-inconsistent sequence $(a_{\mu\wedge i} : i<\lambda)$). Now, by Proposition 2.2.5, we may assume that $(a_{\mu\wedge i} : i<\omega)$ is indiscernible over $(a_v : v\leq\mu)$, for all $\mu\in\omega^{<n-1}$. But that means that $(a_\mu : \mu\in\{0\}^{<n})$ is a k-dividing chain in φ of length n.

4.⇒6.: Let $(a_i : i<\omega)$ be a k-dividing chain of length ω. Let $c\models\wedge_{i<\omega}\varphi(x,a_i)$, and suppose there is an ordinal α, such that $D(p,\varphi,k)<\alpha$ for all types p. Since $\varphi(x,a_i)$ k-divides over $(a_j : j<i)$ for all $i<\omega$, we get $D(c/(a_j : j\leq i),\varphi,k)<D(c/(a_j : j<i),\varphi,k)$ for all $i<\omega$. But this yields an infinite descending sequence of ordinals, a contradiction.

5.$\Rightarrow$1.: We show by induction on n that if $D(p(x),\varphi,k)\geq n$, then there is a (φ,k)-tree of height n consistent with p. This is clearly true for n=0, as both conditions merely say that p is consistent. So suppose that it is true for n, and $D(p,\varphi,k)\geq n+1$, where p is a partial type over A. So there is some tuple b such that $\varphi(x,b)$ k-divides over A, and $D(p(x)\cup\{\varphi(x,b)\},\varphi,k)\geq n$. If n=0, then the single tuple b is a (φ,k)-tree of height 1, consistent with p(x). So suppose that n>0. Then there is a tuple c such that $\varphi(x,c)$ k-divides over $A\cup\{b\}$, and $D(p(x)\cup\{\varphi(x,b),\varphi(x,c)\},\varphi,k)\geq n-1$. By induction hypothesis, there is a (φ,k)-tree $(c_\mu : \mu\in\omega^{<n})$ consistent with $p(x)\cup\{\varphi(x,b),\varphi(x,c)\}$. By the definition of dividing, there is an A-indiscernible sequence $(c_i : i<\omega)$ of type tp(c/Ab) such that $\{\varphi(x,c_i) : i<\omega\}$ is k-inconsistent. Let $(d_{i\wedge\mu} : \mu\in\omega^{<n})$ denote the image of $(c_\mu : \mu\in\omega^{<n})$ under an Ab-automorphism mapping c to c_i. By these ω-many Ab-automorphisms we obtain ω-many (φ,k)-trees of height n, consistent with $p\cup\{\varphi(x,b)\}$. When we join these trees by the root $d_\varnothing$:=b we get the desired (φ,k) tree of height n+1. More exactly: $(d_\mu : \mu\in\omega^{n+1})$ is a (φ,k)-tree of height n+1, consistent with p. Q.e.d.

Corollary 2.4.8 : For every type p, formula $\varphi(x,y)$, $k<\omega$, $n<\omega$, the following are equivalent:

1. $D(p,\varphi(x,y),k)\geq n$
2. There is a (φ,k)-tree of heigth n, consistent with p.

   That is, there is a set of tuples $\{a_v : v\in\omega^{<n}\}$ such that
   - (i) for each $v\in\omega^{<n-1}$, $\{\varphi(x,a_{v\wedge i}) : i<\omega\}$ is k-inconsistent
   - (ii) for every $u\in\omega^n$, $p\cup\{\varphi(x,a_{u\restriction i}) : i<n\}$ is consistent.
3. There is a k-dividing chain of length n in φ, consistent with p.

Proof: 1.$\Rightarrow$2. follows immediately from the proof of Proposition 2.4.7.

2.$\Rightarrow$3. Like the proof of Proposition 2.4.7(2.$\Rightarrow$3.).

3.$\Rightarrow$1. Let $(a_i : i<n)$ be a k-dividing chain of length n in φ, consistent with p. Let $c\models\wedge_{i<n}\varphi(x,a_i)$, and suppose $D(p,\varphi,k)<n$. Since $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all i<n, we get $D(p\cup\{\varphi(x,a_j) : j\leq i\})<D(p\cup\{\varphi(x,a_j) : j<i\}$ for all i<n. This yields an descending sequence of length n, a contradiction.

We would like to give an additional proof for 2.$\Rightarrow$1:

We use induction on n. The case n=0 is clearly true. So suppose there is a (φ,k)-tree of height n+1, consistent with p, witnessed by $\{a_\nu : \nu\in\omega^{<n+1}\}$. Then first, for each i<ω, $\{a_{i\wedge\nu} : \nu\in\omega^{<n}\}$ is a (φ,k)-tree of height n, consistent with $p\cup\{\varphi(x,a_i)\}$, witnessing that $D(p\wedge\varphi(x,a_i),\varphi,k)\geq n$, by the induction hypothesis. And secondly, $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent, by the definition of a (φ,k)-tree. Now, as in the proof of Proposition 2.3.7 (2.⇒3.), we can assume that $(a_i : i<\omega)$ is indiscernible over dom(p). Hence $\varphi(x,a_i)$ k-divides over dom(p) and $D(p,\varphi,n)\geq n+1$.
q.e.d.

Corollary 2.4.9 : If $D(p,\varphi,k)\leq n$, then there is a finite part p_0 of p with $D(p_0,\varphi,k)\leq n$.
D(.,φ,k)=n is a local property, that means, if D(p,φ,k)=n then there exists a finite $p_0\subseteq p$ with $D(p_0,\varphi,k)=n$.

Proof: The second assertion follows from the first. So let us prove the first assertion. Suppose that $D(p_0,\varphi,k)\geq n+1$ for all finite $p_0\subseteq p$. Corollary 2.4.8 and compactness imply $D(p,\varphi,k)\geq n+1$, a contradiction.
q.e.d.

Proposition 2.4.10 : Let φ(x,y) be a formula. Then the following are equivalent:

1. φ has the tree property.
2. φ divides ω_1 times.
3. φ divides α times, for all ordinals α.
4. $D(x=x,\varphi,k)=\infty$, for some k<ω.

Proof: 1.⇒3. follows from Proposition 2.4.7 together with Remark 2.4.5, 3.⇒2. is obvious. So assume that φ divides ω_1 times, witnessed by $(a_j : j<\omega_1)$. Hence there is a function $f:\{a_j : j<\omega_1\}\rightarrow\omega$, with $f(a_j)=k$ if $\varphi(x,a_j)$ k-divides over $\{a_i : i<j\}$. By set theory, f must be constant in ω_1-many arguments, that means that φ k-divides ω_1 times, for some k<ω. Now follows 1. by Proposition 2.4.7.
Since $D(x=x,\varphi,k)\geq D(p,\varphi,k)$ for all partial types p, the equivalency of 2. and 4. follows by the same consideration proving 2.⇒1. and by Proposition 2.4.7.

Q.e.d.

Lemma 2.4.11 : Let q(x) be a partial type over A such that $D(q,\varphi,k)\geq n$, for some $n<\omega$. Then there is a completion $p\in S(A)$ of q with $D(p,\varphi,k)\geq n$.

Proof: We use induction on n; the assertion being trivial for n=0. So suppose it holds, and $D(q,\varphi,k)\geq n+1$. The property $D(.,\varphi,k)\geq n+1$ is (by Corollary 2.4.9) closed under union of chains (that is, if $q_0\subseteq q_1\subseteq\ldots\subseteq q_j\subseteq\ldots$ is a chain of types with $D(q_i,\varphi,k)\geq n+1$, for all $i<\omega$, then $D(\cup_{i<\omega}q_i,\varphi,k)\geq n+1$). Whence, by Zorns's Lemma it is sufficient to show that for any A-formula $\psi(x)$ either $D(q\wedge\psi,\varphi,k)\geq n+1$ or $D(q\wedge\neg\psi,\varphi,k)\geq n+1$.

By definition, there is an A-indiscernible sequence $(b_i : i<\omega)$ such that $D(q\cup\{\varphi(x,b_i)\},\varphi,k)\geq n$, for all $i<\omega$, and $\{\varphi(x,b_i) : i<\omega\}$ is k-inconsistent. By inductive hypothesis, there is a completion $p(x,b_0)$ of $q(x)\cup\{\varphi(x,b_0)\}$ with $D(p,\varphi,k)\geq n$. Let a_i realize $p(x,b_i)$ for $i<\omega$. Then either infinitely many a_i realize ψ, or infinitely many realize $\neg\psi$, according as ψ is an element of infinitely many $p(x,b_i)$ or not. In the first case

$$D(q(x)\cup\{\psi(x),\varphi(x,b_i)\},\varphi,k)\geq D(p(x,b_i),\varphi,k)\geq n$$

for infinitely many $i<\omega$, whence $D(q\cup\{\psi\},\varphi,k)\geq n+1$. In the second case, similarly $D(q\cup\{\neg\psi\},\varphi,k)\geq n+1$.

q.e.d.

Remark 2.4.12 :

(i) Propostion 2.4.11 remains true substituting "≥n" by "=n", since $D(.,\varphi,k)=n+1$ is a local property (again by Corollary 2.4.9) and we can use the same proof with Zorn's Lemma.

(ii) The proof of Proposition 2.4.11does not work for $n\geq\omega$.

Proposition 2.4.13 : If dividing or forking is symmetric, then $D(p,\varphi,k)<\omega$ for all partial types p(x), all formulas $\varphi(x,y)$, and all $k<\omega$.

Hence, if T is a simple theory, then $D(p,\varphi,k)$ is finite for all p, φ, k.

Proof: Suppose $D(x=x,\varphi,k)\geq n$ for all $n<\omega$. By Proposition 2.4.7, φ k-divides λ times for any ordinal λ. That is, there is a k-dividing chain $(a_i : i<\lambda)$ in φ and a tuple b such that $\models\varphi(b,a_i)$ for all $i<\lambda$. Thus, by Proposition 2.2.5 or Ramsey's Theorem and compactness we can find a b-indiscernible sequence $(c_i : i\leq\omega)$ with the same properties ($(c_i : i\leq\omega)$ is a k-dividing chain in φ of length $\omega+1$, and $\models\varphi(b,c_i)$ for all $i\leq\omega$). In particular, $tp(b/c_\omega, c_i : i<\omega)$ divides (forks) over $(c_i : i<\omega)$, since $\varphi(x,c_\omega)\in tp(b/c_\omega, c_i : i<\omega)$. But $tp(c_\omega/b,c_i : i<\omega)$ is finitely satisfiable in $(c_i : i<\omega)$ by b-indiscernibility of $(c_i : i\leq\omega)$, and therefore does not fork (thus, does not divide) over $\{c_i : i<\omega\}$ (Lemma 2.3.5). Hence neither dividing nor forking is symmetric.
q.e.d.

Proposition 2.4.14 : Let T be simple, $A\subseteq B$, and $p\in S(B)$. Then the following are equivalent:

1. p does not fork over A.
2. p does not divide over A.
3. $D(p,\varphi,k)=D(p\lceil A,\varphi,k)$ for all formulas φ and all $k<\omega$.

Proof: 1.⇒2. is trivial.
3.⇒1.: Suppose p forks over A. Then there are $n<\omega$ and formulas $\varphi_i(x,b_i)$ which k_i-divide over A for $i<n$, with $p\vdash\vee_{i<n}\varphi_i(x,b_i)$. Let $\psi(x,y_0,\ldots,y_{n-1},z)$ be the formula $\vee_{i<n}[\varphi_i(x,y_i)\wedge z=y_i]$, and $k=\max\{k_i : i<n\}$. Clearly, we may replace k_i by k, and every $\varphi_i(x,b_i)$ by $\psi(x,c_i)$, where $c_i=b_0\ldots b_{n-1}b_i$, for every $i<n$.
Choose a completion q of p over $B\cup\{c_i : i<n\}$, with $D(q,\psi,k)=D(p,\psi,k)$, (by Remark 2.4.12(i)). Then there is $i_0<n$ such that $q\vdash\psi(x,c_{i0})$, whence

$$D(p\cup\{\psi(x,c_{i0})\},\psi,k)=D(p,\psi,k)$$

$$=D(p\lceil A,\psi,k) \text{ (assuming 3.).}$$

Hence $D(p\lceil A\cup\{\psi(x,c_{i0})\},\psi,k)=D(p\lceil A,\psi,k)$ (this follows by Remark 2.4.3(ii)).
As $\psi(x,c_{i0})$ k-divides over A, we get $D(p\lceil A,\psi,k)\geq D(p\lceil A,\psi,k)+1$, a contradiction.
2.⇒3. Suppose p does not divide over A. We shall show inductively on n that $D(p\lceil A,\varphi,k)\geq n$ implies $D(p,\varphi,k)\geq n$. From this follows $D(p\lceil A,\varphi,k)=D(p,\varphi,k)$.
This is clearly true for n=0 by consistency of p, so assume it holds for n, and $D(p\lceil A,\varphi,k)\geq n+1$. Let $(b_i : i<\omega)$ be an A-indiscernible sequence such that

$D(p\lceil A\cup\{\varphi(x,b_i)\},\varphi,k)\geq n$ for all $i<\omega$, and $\{\varphi(x,b_i) : i<\omega\}$ is k-inconsistent. By Lemma 2.4.11 there is a completion $q\in S(Ab_0)$ of $p\lceil A\cup\{\varphi(x,b_0)\}$ with $D(q,\varphi,k)\geq n$. If $c\models q$ and $d\models p$, then there is an A-automorphism f with f(c)=d. This f moves q to some $q'\in S(Af(b_0))$ such that $D(q',\varphi,k)=D(q,\varphi,k)$ (rank is invarinat under automotphisms) and $q'\cup p$ is consistent (namely realized by d). So we can assume without lost of generality that there is a realization a of $q\cup p$. By 3.⇒1. trivially $tp(b_0/Aa)$ does not fork over Aa, and has a non-forking extension to Ba by Lemma 2.3.3. Conjugating over Aa, we may assume that $tp(b_0/Ba)$ does not fork over Aa, whence $b_0\perp_{Aa}B$. As $a\perp_A B$ by assumption, Lemma 2.3.7 yields $b_0a\perp_A B$, whence $a\perp_{Ab0}B$ by symmetry and Lemma 2.3.2.8. By inductive hypothesis,

$$D(a/Ab_0,\varphi,k)=D(q,\varphi,k)\geq n$$

implies $D(a/Bb_0,\varphi,k)\geq n$. Finally, since $B\perp_A b_0$ by symmetry, there is a B-indiscernible A_{b0}-conjugate $(b_i' : i<\omega)$ of $(b_i : i<\omega)$ by Proposition 2.3.6, witnessing $D(a/B,\varphi,k)\geq D(a/Bb_0,\varphi,k)+1\geq n+1$.
q.e.d.

Remark 2.4.15 : Note that the implication 3.⇒1. needs simplicity only in the form $D(p,\varphi,k)<\omega$ for all formulas φ and all $k<\omega$. Furthermore, it also works for a partial type r instead of p.

Definition 2.4.16 : A formula φ(x,y) has the *strict order property* if it defines a partial order with arbitrarily long chains.
A theory T has the *strict order property* if some formula has it in some model of T.

Lemma 2.4.17 : A simple theory does not have the strict order property.

Proof: Suppose φ(x,y) is a formula with the strict order property. By compactness and Ramsey's Theorem, there is a model of T where φ orders an indiscernible chain $(a_i : i\in Q)$, where Q denotes the set of the rational numbers.
Let $\psi(x,yy')$ be the formula $\varphi(y,x)\wedge\varphi(x,y')$. Let $i,j\in Q$, $i<j$, and $k_0=j$, $k_{n+1}=(i+k_n)/2$. Clearly, the set $\{\psi(x,a_ia_{k_n}) : n<\omega\}$ is consistent, but $\psi(x,a_ia_j)\wedge\psi(x,a_sa_t)$ is inconsistent for $i<j\leq s<t$. So

it is easy to see that there exists an infinite dividing chain in ψ, hence $D(x=x,\psi(x,yy'),2)=D(x=x,\varphi(y,x)\wedge\varphi(x,y'),2)$ is infinite, and T is not simple by Proposition 2.4.13.
q.e.d.

We shall now collect the properties of non-forking (independence) in a simple theory.

Theorem 2.4.18 : **Properties of Independence**
Suppose T is simple, and $A\subseteq B\subseteq C$. Then the following hold:

1. **Existence** If $p\in S(A)$, then p does not fork over A.
2. **Extension** Every partial type over B which does not fork over A has a completion over B which does not fork over A.
3. **Reflexivity** $B\perp_A B$ if and only if $B\subseteq acl(A)$.
4. **Monotonicity** If p and q are types with $p\vdash q$ and p does not fork over A, then q does not fork over A.
5. **Finite Character** $D\perp_A B$ if and only if $d\perp_A B$ for every finite tuple $d\in D$.
6. **Symmetry** $D\perp_A B$ if and only if $B\perp_A D$.
7. **Transitivity** $D\perp_A C$ if and only if $D\perp_A B$ and $D\perp_B C$.
8. **Local Character** For any $p\in S(A)$ there is $A_0\subseteq A$ with $card(A_0)\leq card(T)$, such that p does not fork over A_0.

Proof:
7.: Left to right is Lemma 2.3.2. For the other direction let $d\in D$ and suppose $d\perp_A B$ and $d\perp_B C$. Then $D(d/C,\varphi,k)=D(d/B,\varphi,k)=D(d/A,\varphi,k)$ for all formulas φ and all $k<\omega$, by Proposition 2.4.14. Therefore $d\perp_A C$ for all $d\in D$, again by Proposition 2.4.14, whence $D\perp_A C$.
8.: Consider $p\in S(A)$. For every formula φ and every $k<\omega$ there is a finite $a_{\varphi,k}\in A$ with $D(p,\varphi,k)=D(p\lceil a_{\varphi,k},\varphi,k)$ by Corollary 2.4.9. If A_0 is the union of all these $a_{\varphi,k}$ for all formulas φ and all $k<\omega$, then $D(p,\varphi,k)=D(p\lceil A_0,\varphi,k)$. for all φ, and all $k<\omega$. So p does not fork over A_0 by Proposition 2.4.14, and clearly, $card(A_0)\leq card(T)$.

The others assertions are obvious by the equivalence of dividing and forking and by Lemma 2.3.2 and Lemma 2.3.3.
q.e.d.

Note that there are examples of theories in which dividing and forking is the same, but independence is not symmetric. Hence, simplicity is not implied by the equivalence of dividing and forking.

The following is often useful:

Proposition 2.4.19 : Let T be simple.

(i) Let $B \subseteq C$ and $A \perp_B C$. Then for any set D with $C \subseteq D$ there is A' such that tp(A'/C)=tp(A/C) and $A' \perp_C D$.

(ii) Let A, B, C be sets with $B \subseteq C$. There is A' such that tp(A'/B)=tp(A/B) and $A' \perp_B C$.

(iii) $A \perp_D B$ and $A \perp_D C$ if and only if $A \perp_D BC$.

Proof: (i) follows from Theorem 2.4.18.2, and (ii) follows from Theorem 2.4.18.1. and (i). Right to left of (iii) is clear. The other direction: From $A \perp_D C$ follows trivially $A \perp_{DB} C$. Since $A \perp_D B$ we can use transitivity of forking (Theorem 2.4.18.7) with $D \subseteq DB \subseteq DBC$ to obtain $A \perp_D C$.
q.e.d.

Lemma 2.4.20 : If T is simple, then for every complete type $p(x) \in S(A)$ and every partial type $\Phi(x,y)$ there is a partial type q(y) such that for any tuple a there is a non-forking extension of p to Aa containing $\Phi(x,a)$ if and only if $\models q(a)$.

Proof : If $D(p,\varphi,k)=n(\varphi,k)$, then q(y) is the partial type expressing $D(p(x) \cup \Phi(x,y),\varphi,k) \geq n(\varphi,k)$ for all formulas φ and all $k<\omega$. (see Remark 2.4.3)
q.e.d.

Definition 2.4.21 : A type $p \in S(A)$ is *stationary* if it has only one non-forking extension to any superset of A.

Corollary 2.4.22 : If $q \in S(A)$ ia a stationary type in a simple theory, then for every formula $\varphi(x,y)$ there is a formula $\psi(y)$ over A, such that $\models \psi(b)$ if and only if $\varphi(x,b)$ is in the (unique) non-forking extension of q to Ab.

Proof : By Lemma 2.4.20 there are partial types p and p' such that $\models p(b)$ if and only if $\varphi(x,b)$ is in the non-forking extension of q to Ab, and $\models p'(b)$ if and only if $\neg\varphi(x,b)$ is in the non-forking extension of q to Ab. Since q is stationary, exactly one of the two cases must hold. Hence we get $\models p(y) \wedge p'(y)$ is inconsistent, and for all tuples b: $\models p(b)$ or $\models p'(b)$. Compactness implies that we can replace p and p' by finite subtypes, i.e. formulas χ and χ'. Then $\psi(y)=\chi(y) \wedge \neg\chi'(y)$, or equivalently $\psi(y)=\chi(y)$.
q.e.d.

Definition 2.4.23 : Let $\varphi(x,y)$ be a formula. A type $p(x) \in S(A)$ is *φ-definable* over B if there is a formula $d_p\varphi(y)$ over B such that for any $a \in A$ we have $\models d_p\varphi(a)$ if and only if $\varphi(x,a) \in p$. We call $d_p\varphi$ a φ-definition for p over B; it is also denoted as $d_p x\varphi(x,y)$. Finally, p is *definable* if it is φ-definable over A for all formulas φ; a *defining scheme* for p is the collection of its φ-definitons, for all formulas φ.

Remark 2.4.24 : A stationary type in a simple theory has a unique canonical defining scheme (up to equivalence), namely the defining scheme for all its non forking extensions given by Corollary 2.4.22.

Theorem 2.4.25 : Let **M** be a model of a simple theory, $A \subseteq \mathbf{M}$ and $p \in S(\mathbf{M})$.
If p is definable over A, then p does not fork over A.
If $p \upharpoonright A$ is stationary and p does not fork over A, then p is definable over A.

Proof : Suppose for every formula there is a φ-definition $d_p\varphi$ over A. For a formula $\varphi(x,b)\in p$ let $(b_i : i<n)$ be a sequence of type tp(b/A) in **M**. Then $\mathbf{M} \models d_p\varphi(b_i)$ for all $i<n$, so $\cup_{i<n}\varphi(x,b_i)$ is a subset of p and thus consistent. It follows that $\varphi(x,b)$ does not fork over A, and neither does p. q.e.d.

2.5 Morley Sequences

B. Kim used in his thesis [Kim1] an important technical tool to prove the equivalence of dividing and forking, namely Morley sequences. The original definition by Shelah [Sh2] (and in Kim's thesis too) of a simple theory was in terms of finiteness of the local D(.,φ,k)-ranks. In this section, we shall prove that symmetry of independence, and hence simplicity, is equivalent to a number of conditions. To this end we need the notion of a Morley sequence.

2.5.1 Definition : A sequence $(a_i : i\in I)$ is *independent over A*, or *A-independent*, if

$$a_i \perp_A (a_j : j<i) \text{ for all } i\in I.$$

Remark : Suppose T is simple.Let I be an ordered index set and $(a_i : i\in I)$ an A-independent sequence. Then $a_i \perp_A (a_j : j\neq i)$ for any $i\in I$.

Proof : Suppose that $a_i \not\perp_A (a_j : j\neq i)$. By symmetry $(a_j : j\neq i) \not\perp_A a_i$, so by Finite Character there is a minimal finite set $J\subseteq I-\{i\}$ such that $(a_j : j\in J) \not\perp_A a_i$. If $j<i$ for all $i\in J$, symmetry yields a contradiction to our assumption; otherwise let k be the maximal element of J. So $(a_j : j\in J, j\neq k) \perp_A a_i$, by minimality of J; as $a_k \perp_A a_i$ and thus trivially $a_k \perp_{A(aj\ :\ j\in J,\ j\neq k)} a_i$, symmetry and transitivity yield $a_i \perp_A (a_j : j\in J)$, a contradiction.
q.e.d.

2.5.2 Definition : Let $A\subseteq B$ and $p\in S(B)$. A *Morley sequence* in p over A is a B-indiscernible sequence $(a_i : i<\omega)$ of realizations of p, such that $tp(a_i/Ba_0\ldots a_{i-1})$ does not fork over A, for all $i<\omega$.
If A=B, a Morley sequence in p over A is simply called a Morley sequence in p.

Thus, a Morley sequence in $p \in S(A)$ is an A-indiscernible, A-independent sequence of realizations of p. We may occasionally index Morley sequences by infinite ordered sets other than ω.

Lemma 2.5.3 : If $p \in S(B)$ does not fork over $A \subseteq B$, then there is a Morley sequence in p over A. (T is not necessarily simple.)

Proof : Suppose for some ordinal α we have found a sequence $(a_i : i<\alpha)$ of realisations of p, such that $a_i \perp_A B(a_j : j<i)$ for all $i<\alpha$ and $tp(a_i/B,a_j : j<i)=tp(a_k/B,a_j : j<i)$ whenever $i \leq k<\alpha$. (Since p does not fork over A, the existence of a sequence of length $\alpha=1$ with this property is guaranteed by choosing some realisation of p.) By Lemma 2.3.3 there is some $a_\alpha \models \cup_{i<\alpha} tp(a_i/B,a_j : j<i)$ such that $tp(a_\alpha/B,a_i : i<\alpha)$ does not fork over A. So we can find such a sequence of arbitrary length α. Let $\kappa=card(A)+card(T)$, $\mu=(2^\kappa)^+$, and let $\alpha=\Im_\mu$. By Proposition 2.2.5 we can find a sequence $(b_i : i<\omega)$ of realisations of p, indiscernible over B and A-independent, hence a Morley sequence in p over A.
Q.e.d.

Corollary 2.5.4 : Every complete type in a simple theory has a Morley sequence (of arbitrary length).

Proof : Since tp(a/A) does not fork over A, in every simple theory, the assertion follows from Lemma 2.5.3 with A=B.
q.e.d.

The following assertion yields Morley sequences in any theory.

Definitition 2.5.5 : Let **M** be a model, $\mathbf{M} \subseteq A \subseteq B$, and $p \in S(B)$ a coheir of $p \lceil \mathbf{M}$. Suppose $(a_i : i<\omega)$ is a sequence of tuples of B such that $a_i \models p \lceil A \cup (a_j : j<i)$. Then $(a_i : i<\omega)$ is called a *coheir sequence in p over* (**M**,A). If A=**M**, it is omitted.

Lemma 2.5.6 : Let **M**, A, B be as in Definition 2.5.5, and suppose $I=(a_i : i<\omega)$ is a coheir sequence in p over (**M**,A). Then I is a Morley sequence in $p\lceil A$ over **M**.

Proof : As $tp(a_i/Aa_j : j<i)$ is finitely satisfiable in **M**, for all $i<\omega$, it does not fork over **M**, hence I is M-independent.
For indiscernibility, we show by induction on $n<\omega$ that $tp(a_0\ldots a_n/A)=tp(a_{i0}\ldots a_{in}/A)$ for all $i_0<\ldots<i_n$. This is trivial for n=0. Suppose it holds for n-1. Then $tp((a_j : j<n)/A)=tp((a_{ij} : j<n)/A)$, whence $tp(A,a_0\ldots a_{n-1}a_{in}/\mathbf{M})=tp(A,a_{i0}\ldots a_{in-1}a_{in}/\mathbf{M})$ by Lemma 2.3.5. Therefore $tp(a_0\ldots a_{n-1}a_n/A)=tp(a_{i0}\ldots a_{in-1}a_{in}/A)$. As a_{in} realises $tp(a_i/Aa_j : j<i)$ for all $i_n\geq i$, considering the corresponding $(Aa_j : j<i)$-automorphisms we obtain $tp(a_0\ldots a_{n-1}a_n/A)=tp(a_0\ldots a_{n-1}a_{in}/A)$, and the assertion follows.
q.e.d.

Theorem 2.5.7 : The following conditions are equivalent:

1. $D(x=x,\varphi,k)=\infty$, for some $k<\omega$.
2. There is an indiscernible sequence $(c_ia_i : i<\omega)$ such that for every $i<\omega$ holds: $\models\varphi(c_i,a_0)$, and $\varphi(x,a_i)$ k-divides over $\{c_ja_j : j<i\}$, for some $k<\omega$.
3. There is a tuple c and a c-indiscernible sequence $(a_i : i<\omega)$ such that $\models\varphi(c,a_0)$, and for every $i<\omega$ holds: $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$, for some $k<\omega$.
4. There is a sequence $(a_i : i\leq\omega)$ of $tp(a_0)$ such that $\models\varphi(x,a_\omega)$ divides over $\{a_i : i<\omega)$, and $\{\varphi(x,a_i) : i\leq\omega\}$ is consistent.

Proof : 1.⇒2.: By Corollary 2.4.8 and compactness, 1. implies the existence of a k-dividing chain $(b_i : i<\omega)$ in φ of length ω. Now we construct inductively the sequence $(c_ia_i : i<\omega)$ such that $\varphi(x,a_i)$ k-divides over $\{c_ja_j : j<i\}$, and $\models\varphi(c_i,a_0)\wedge\ldots\wedge\varphi(c_i,a_i)$. Then, by compactness, we can find such a sequence of arbitrary length λ. Using Proposition 2.2.5, we then may assume that the original sequence is indiscernible.
We start the construction by choosing $a_0=b_0$ and c_0 such that $\models\varphi(c_0,a_0)$. $\varphi(x,b_1)$ k-divides over $a_0=b_0$, witnessed by some a_0-indiscernible sequence I_1 . By the remark after Proposition 2.2.5,

there is an a_0c_0-indiscernible sequence I_1', a_0 conjugated to I_1 (written $I_1' \equiv_{a0} I_1$). Hence I_1' witnesses that $\varphi(x,b_1')$ k-divides over a_0c_0, where b_1' is chosen such that

$$(b_i' : i<\omega)I_1' \equiv_{a0} (b_i : i<\omega)I_1,$$

(i.e., b_1' is the image of b_1 under the a_0-automorphism mapping I_1 to I_1'). Choose $a_1=b_1'$ and c_1 such that $c_1 \models \varphi(x,a_0)\wedge\varphi(x,a_1)$. Now it is clear how to continue the construction: Since $\varphi(x,b_2')$ k-divides over (a_0,a_1), witnessed by some sequence I_2, again by Proposition 2.2.5, there is a sequence I_2' witnessing that $\varphi(x,b_2'')$ k-divides over $(a_0c_0a_1c_1)$, where $(b_i'' : i<\omega)$ is the image of $(b_i' : i<\omega)$ under the a_0a_1-automorphism mapping I_2 to I_2'.
Then choose $a_2=b_2''$ and c_2 such that $c_2 \models \varphi(x,a_0)\wedge\varphi(x,a_1)\wedge\varphi(x,a_2)$.
Repeating the argument ω times, we obtain the desired sequence.
2.⇒3.: We can apply compactness to produce a sequence as in 2. of arbitrary length, in particular of length $\omega+1$. Under this assumption put $c=c_\omega$. Then $(a_i : i<\omega)$ is c-indiscernible, $\models\varphi(c,a_0)$, and $\varphi(x,a_i)$ k-divides over $(a_j : j<i)$ for all $i<\omega$.
3.⇒4.: By compactness, there is such a sequence as in 3. of length $\omega+1$.
4.⇒1. Suppose $\varphi(x,a_\omega)$ k-divides over $\{a_i : i<\omega\}$ for some k. Then $\varphi(x,a_\omega)$ k-divides over $\{a_j : j<i\}$ for all $i<\omega$. Since $tp(a_\omega)=tp(a_i)$, $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all $i<\omega$. Then compactness yields a k-dividing chain in φ of arbitrary length. Hence $D(x=x,\varphi,k)=\infty$, by Proposition 2.4.10.
q.e.d.

Remark 2.5.8 : By compactness it is clear that we can substitute in the assertions 2. and 3. of the previous Theorem ω by any infinite linear ordered index set.

Lemma 2.5.9 : Let $I=(a_i : i<\omega)$ be a Morley sequence over A, and let $(a_0^j,\ldots,a_n^j : j<\omega)$ be an A-indiscernible sequence in $tp(a_0,\ldots,a_n/A)$. Then there is an A-automorphic image J of I such that:

1. $(a_0^j,\ldots,a_n^j)\wedge J$ is indiscernible over A, for all $j<\omega$
2. $(a_0^j,\ldots,a_n^j : j<\omega)$ is indiscernible over AJ.

Proof : We shall inductively find b_i for $i<\omega$ such that

(i) $tp(a_0^j,\ldots,a_n^j,b_0,\ldots,b_{m-1})=tp(a_0,\ldots,a_n,a_{n+1},\ldots,a_{n+m})$, and

(ii) $(a_0^j,\ldots,a_n^j : j<\omega)$ is indiscernible over $A\cup\{b_i : i<m\}$.

So assume we have already found those b_i for $i<m$. Put

$$p(x,a_i : i\leq n+m)=tp(a_{n+m+1}/A,a_i : i\leq n+m).$$

This type does not divide over A. Since $(a_0^j,\ldots,a_n^j,b_0,\ldots,b_{m-1} : j<\omega)$ is an A-indiscernible sequence (by (ii)) in $tp(a_i : i\leq n+m/A)$ (by (i)), Proposition 2.3.6 implies that there is some b_m realizing

$$\cup_{j<\omega}p(x,a_0^j,\ldots,a_n^j,b_0,\ldots,b_{m-1}),$$

such that the sequence remains indiscernible over Ab_m. Then b_m will have the necessary properties.

It is clear that the sequence $J=(b_i : i<\omega)$ is the required A-conjugate of I.

q.e.d.

Theorem 2.5.10 : The following is equivalent:

1. T is simple.
2. Forking (dividing) satisfies symmetry.
3. Forking (dividing) satisfies transitivity.
4. Forking (dividing) satisfies local character.
5. $D(p,\varphi,k)<\omega$ for all types p, all formulas φ, and all $k<\omega$.
6. A formula $\varphi(x,a)$ does not fork (divide) over A if and only if for some Morley sequence I in $tp(a/A)$ the set $\{\varphi(x,a') : a'\in I\}$ is consistent.
7. A formula $\varphi(x,a)$ does not fork (divide) over A if and only if for any Morley sequence I in $tp(a/A)$ the set $\{\varphi(x,a') : a'\in I\}$ is consistent.

Proof : 1. implies 2., 3. and 4. by Theorem 2.4.18. The implication 2.⇒5. is Proposition 2.4.13.

3.⇒5.: Consider the set $\omega+2+\omega^*$, where $\omega^*=\{-1,-2,\ldots\}$. This set is of order type

$$0<1<\ldots<\omega<\omega+1<\ldots<-2<-1.$$

If $D(p,\varphi,k)\geq\omega$, that is $=\infty$, then by Lemma 2.5.7 and Remark 2.5.8 there is an indiscernible sequence $(c_ia_i : i<\omega+2+\omega^*)$, such that for every i, $\sim\varphi(c_i,a_0)$ and $\varphi(x,a_i)$ divides over $\{c_ja_j :$

$j<i\}$. Let $I=\{c_i : i<\omega\}$ and let $J=\{c_i : i\in\omega^*\}$. By indiscernibility, $\sim\varphi(c_{\omega+1},a_\omega)$, hence $tp(c_{\omega+1}/IJa_\omega\}$ divides over I. But (by indiscernibility) $tp(c_{\omega+1}/IJ)$ is finitely satisfiable in I, hence does not fork over I, and $tp(c_{\omega+1}/IJa_\omega)$ is finitely satisfiable in J, hence does not fork over J (see Lemma 2.3.5). Thus, transitivity of dividing and of forking fails.

4.⇒5.: If $D(p,\varphi.k)\geq\omega$, then by Lemma 2.5.7 there is a tuple c and a c-indiscernible sequence $(a_i : i<card(T)^+)$ such that $\sim\varphi(c,a_0)$, and $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all $i<card(T)^+$. If $A=\{a_i : i<card(T)^+\}$ and $A_0\subseteq A$ with $card(A_0)\leq card(T)$, then there is $i<card(T)^+$ such that $A_0\subseteq\{a_j : j<i\}$. However, $\varphi(x,a_i)\in tp(c/A)$ witnesses that this type divides, and hence forks, over A_0.

6.⇒5.: Again let ω^* be the set $\{-1,-2,\ldots\}$. If $D(p,\varphi,k)\geq\omega$, then by Lemma 2.5.7 there is a tuple c and a c-indiscernible sequence $(a_i : i<\omega+\omega^*)$ such that $\sim\varphi(c,a_0)$, and $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all $i\in\omega+\omega^*$. Put $A=\{a_i : i<\omega\}$. As $tp(a_i/A(a_j : j<i)$, for every $i\in\omega^*$, is finitely satisfiable in A (by indiscernibility) and thus does not fork over A (Lemma 2.3.5), the sequence $(a_i : i\in\omega^*)$ is a Morley sequence in $tp(a_{-1}/A)$. However, $\varphi(x,a_{-1})$ divides (and hence forks) over A, while $\wedge_{i\in\omega^*}\varphi(x,a_i)$ is consistent.

5.⇒6.: Assume $D(p,\varphi,k)<\omega$ for all types p, all formulas φ, and all $k<\omega$, and suppose $\varphi(x,a)$ divides over A. Let I be a Morley sequence in tp(a/A). As $\varphi(x,a)$ divides over A, there is an A-indiscernible sequence $(a_i : i<\omega)$ such that $\wedge_{i<\omega}\varphi(x,a_i)$ is k-inconsistent for some $k<\omega$. By Lemma 2.5.9 we may assume that $a_j\wedge I$ is A-conjugate to I (and A-indiscernible), for all $j<\omega$, and $(a_j : j<\omega)$ is indiscernible over AI.

Put $p(x)=\wedge_{a'\in I}\varphi(x,a')$, and suppose p is consistent. Then the sequence $(a_i : i<\omega)$ witnesses that $D(p(x)\cup\{\varphi(x,a_0)\},\varphi,k)<D(p,\varphi,k)$. But these two partial types are A-conjugated and must have the same $D(.,\varphi,k)$-rank, a contradiction. Therefore p is not consistent.

For the converse, note that since $D(a/A,\varphi,k)<\omega$ for all formulas φ and all $k<\omega$, Remark 2.4.15 implies that tp(a/A) does not fork over A. By Lemma 2.5.3 there is a Morley sequence I in tp(a/A). By definition, if $\varphi(x,a)$ does not divide over A, then $\{\varphi(x,a') : a'\in I\}$ is consistent.

Next, to prove the direction from right to left of 6. also in the case of forking, we show that forking is the same as dividing. So let $\varphi(x,a)$ be a formula which forks over A. So there are $n<\omega$ and $\psi_i(x,b_i)$ for $i<n$ such that $\varphi(x,a)\vdash\vee_{i<n}\psi_i(x,b_i)$, and each $\psi_i(x,b_i)$ divides over A.

Adding dummy variables, we may assume $a=b_0=\ldots=b_{n-1}$. So we obtain $\varphi(x,a)\vdash\vee_{i<n}\psi_i(x,a)$. Let $(a_j : j<\omega)$ be a Morley sequence in tp(a/A). If $\varphi(x,a)$ does not divide over A, then $\wedge_{j<\omega}\varphi(x,a_j)$ is consistent, say it is realized by b. Hence, $\varphi(x,a_j)\vdash\vee_{i<n}\psi_i(x,a_j)$, for all $j<\omega$. But then there is $i_0<n$ and an infinite subset $I\subseteq\omega$ such that b realizes $\psi_{i0}(x,a_j)$, for all $i\in I$. As $(a_i : i\in I)$ is also a Morley sequence in tp(a/A), the formula $\psi_{i0}(x,a)$ cannot divide over A, a contradiction. Thus, $\varphi(x,a)$ divides over A, and hence, forking implies dividing.

5.⇒7. and 7.⇒5. is proved as the equivalence of 5. and 6.

5.⇒1.: By the last part, forking is the same as dividing, and 6. holds.

We show that our notion of independence $\perp$ satisfies symmetry. So suppose tp(a/Ab) does not fork over A, $(a\perp_A b)$. By Lemma 2.5.3 there is a Morley sequence $(a_i : i<\omega)$ in tp(a/Ab) over A. By partial transitivity, Lemma 2.3.2, this is also a Morley sequence in tp(a/A). But if $\varphi(x,a,A)\in tp(b/Aa)$, then by Ab-indiscernibility of the Morley sequence $\models\varphi(b,a_i,A)$ for all $i<\omega$. Therefore $\varphi(x,a,A)$ does not fork over A, by 6. Thus, tp(b/Aa) does not fork over A, $(b\perp_A a)$, and symmetry of independence is satisfied. Hence, T is simple.

q.e.d.

Now we can give a further characterization:

Proposition 2.5.11 : The following conditions are equivalent:

1. T is simple.
2. There are no tuple b and a sequence $(a_i : i<card(T)^+)$ such that $tp(b/a_j : j\leq i)$ divides over $(a_j : j<i)$, for all $i<\omega_1$.
3. No formula in T has the tree property.
4. No formula in T divides ω_1 times.

Proof :

1.⇒2. follows from local character of forking.

2.⇒3.: Suppose $\varphi(x,y)$ has the tree property. Then, by Proposition 2.4.10, φ divides ω_1 times. Hence, there are a tuple b and types containing φ as in 2.

3.⇒1.: If T is not simple, then $D(x=x,\varphi,k)\geq\omega$, for some φ, k (by Theorem 2.5.10). Then Proposition 2.4.7 says that $\varphi(x,y)$ has the tree property.
The equivalence of 3. and 4. is Proposition 2.4.10.
q.e.d.

Definition 2.5.12 : A simple theory is called *supersimple* if no type divides over all finite subsets of its domain.
(That is, if p is a type over the set A, then there is a finite subset $A_0\subseteq A$ such that p does not divide over A_0.)

Proposition 2.5.13 : The following is equivalent:

1. T is supersimple.
2. There are no b and a sequence $(a_i : i<\omega)$ such that $tp(b/a_j : j\leq i)$ divides over $(a_j : j<i)$ for all $i<\omega$.

Proof :
1.⇒2.: Suppose ¬2. Then $tp(b/(a_i : i<\omega))$ divides over any finite subset of its domain, hence we get ¬1.
2.⇒1.: Suppose that T is not supersimple, and p is a type over A which divides over any finite subset of A. Then A must be infinite. Let b be some realization of p. Choose some $a_0\in A$. Then tp(b/A) divides over $\emptyset$. But there is some finite $a_0\in A$ such that $tp(b/a_0)$ divides over $\emptyset$. Now we find some $a_1\in A$, such that $tp(b/a_0a_1)$ divides over a_0, and so on. So we can proceed ω times to produce a sequence $(a_i : i<\omega)$ of elements of A which satisfies the negation of 2.
q.e.d.

Remark 2.5.14 : If a formula $\varphi(x,y)$ divides ω times, then there are b and a sequence $(a_i : i<\omega)$ such that $\models\varphi(b,a_i)$, for all $i<\omega$, and $\varphi(x,a_i)$ divides over $(a_j : j<i)$. Hence, the type $\{\varphi(x,a_i) : i<\omega\}$ divides over any finite subset of its domain, and T can not be supersimple. Therefore, if T is supersimple, then no formula divides ω times.

2.6 The Independence Theorem (Amalgamation of types)

In this section T will be a simple theory.

Lemma 2.6.1 : Let (a_i : i<ω+ω) be an A-indiscernible sequence, and put I=(a_i : i<ω) and I'=($a_{\omega+i}$: i<ω). Then I' is a Morley sequence in tp(a_ω/AI)

Proof: I' is clearly an AI-indiscernible sequence of realizations of tp(a_ω/AI). We have to check that it is independent. tp($a_{\omega+i}$/AI$a_{\omega+j}$: j>i) does not fork over AI for all i<ω, since it is finitely satisfiable in I (see Lemma 2.3.5). Then for every i<ω holds:

(*) $a_{\omega+I} \perp_{AI} a_{\omega+j}$, for all j>i.

Now let j<ω. Then by (*) and symmetry: $a_{\omega+j} \perp_{AI} a_\omega$, $a_{\omega+j} \perp_{AI} a_{\omega+1}$, … , $a_{\omega+j} \perp_{AI} a_{\omega+j-1}$. By a generalization of Proposition 2.4.19.3, we get $a_{\omega+j} \perp_{AI} (a_{\omega+i}$: i<j). Since j was arbitrary, this holds for all j<ω, and I' is independent over AI.
q.e.d.

Proposition 2.6.2 : Let p(x,a) be a partial type over Aa which does not fork over A. If (a_i : i<ω) is a Morley sequence over A in tp(a/A), then q=$\cup_{i<\omega}$p(x,a_i) is consistent and does not fork over A.

Proof : q is consistent by Proposition 2.3.6. Since T is simple, forking is the same as dividing. Suppose bΓq, and φ(x,a_i : i<n)∈q(x) (where we have supressed possible parameters from A). Put b_j=(a_{jn+i} : i<n), then (b_j : j<ω) is also an infinite A-indiscernible, A-independent sequence, and $\wedge_{j<\omega}$φ(x,b_j) is realized by b, whence consistent. Hence φ(x,b_0)=φ(x,a_i : i<n) does not fork over A, by Theorem 2.5.10.6. If q was fork over A there would be a formula ψ which is implied by q and forks over A. Then ψ would be implied by some φ∈q which must fork over A too, a contradiction.

q.e.d.

Theorem 2.6.3 : Let $p(x,a)$ be a partial type over Aa which does not fork over A. If $(a_i : <\omega)$ is indiscernible over A with $tp(a_0/A)=tp(a/A)$, then $q=\cup_{i<\omega}p(x,a_i)$ is consistent and does not fork over A.

Proof : Let I be a sequence of order type ω such that $I\wedge(a_i : i<\omega)$ is indiscernible over A, and let $r(x,a_0)$ be a completion of $p(x,a_0)$ to AIa_0 which does not fork over A. Then $(a_i : i<\omega)$ is a Morley sequence in $tp(a_0/AI)$ by Lemma 2.6.1; since $r(x,a_0)$ does not fork over AI, the set $\cup_{i<\omega}r(x,a_i)$ is consistent and does not fork over AI, by Proposition 2.6.2. Its restriction to AI is equal to $r(x,a_0)\lceil_{AI}$ which does not fork over A. By transitivity, $\cup_{i<\omega}r(x,a_i)$ does not fork over A. Hence, $\cup_{i<\omega}p(x,a_i)\subseteq\cup_{i<\omega}r(x,a_i)$ does not fork over A.
q.e.d.

Definition 2.6.4 : A relation $R(x,y)$ (which need not be definable) is *A-invariant* if it is invariant under all automorphisms fixing A pointwise. That is, for every $f\in Aut(\mathbf{C}/A)$ and every tuple a

$$a\in R \text{ if and only if } f(a)\in R.$$

If $A=\emptyset$, we say that R is *invariant.*
(Note that R is invariant in the theory T if and only if R is invariant in the theory T(A).)
An A-invariant relation is *stable* if there is no A-indiscernible sequence $(a_ib_i : i<\omega)$ such that $R(a_i,b_j)$ holds if and only if $i\leq j$.

Remark 2.6.5 : Instead of the condition $i\leq j$ we could equally well have put $i<j$, $i>j$, or $i\geq j$.

Lemma 2.6.6 : Let $p(x,y)$ and $q(x,z)$ be partial types. Then the relation "$p(x,a)\wedge q(x,b)$ does not fork over A" is a stable relation.

Proof : Let $(a_ib_i : i<\omega)$ be an A-indiscernible sequence such that $p(x,a_i)\wedge q(x,b_j)$ forks over A if and only if $i>j$. In particular $p(x,a_0)\wedge q(y,b_0)$ does not fork over A. But then by Theorem 2.6.3

the whole set $\cup_{i<\omega}p(x,a_i)\wedge q(x,b_i)$ does not fork over A, and neither does $p(x,a_i)\wedge q(x,b_j)$, for any $i,j<\omega$, by indiscernibilty.

Lemma 2.6.7 : Let R be an A-invariant relation. Suppose there is an A-indiscernible sequence $(b_i : i\in I)$, some $i_0\in I$ and some a, such that

1. $tp(ab_i/A)$ is constant for all $i\leq i_0$, and it is constant for all $i>i_0$
2. $R(a,b_i)$ holds if and only if $i\leq i_0$
3. both $\{i\in I : i\leq i_0\}$ and $\{i\in I : i>i_0\}$ are infinite.

Then R is unstable.

Proof : By compactness, we may assume that I is the set Z of integers and $i_0=0$. By indiscernibility, we find a_i for $i\in I$ such that

$$(*) \qquad tp((a_i,b_{j-i} : j\in I)/A)=tp((a,b_j : j\in I)/A).$$

By Ramsey's Theorem we find an infinite subset $J\subseteq I$ such that $(a_jb_j : j\in J)$ is A-indiscernible. By A-indiscernibility of $(b_i : i\in I)$ there is an A-automorphism mapping $(a_jb_j : j\in J)$ onto $(a_i'b_i : i\in I)$, for some $(a_i' : i\in I)$, and $(a_i'b_i : i\in I)$ is A-indiscernible and satisfies (*). Hence, we may assume that the sequence $(a_ib_i : i\in I)$ is indiscernible over A.
But then $R(a_i,b_j)$ holds if and only if $R(a_0,b_k)$ holds (for some k, with $i<j$ if and only if $0<k$, by indiscernibility), if and only if $R(a,b_k)$ holds (by (*)), if and only if $k\leq 0$ (by 2.), if and only if $j\leq i$. Thus, $R(a_i,b_j)$ holds if and only if $j\leq i$, and R is unstable.
q.e.d.

Theorem 2.6.8 : Suppose **M** is a model of T and R is an **M**-invariant stable relation such that R(a,b) holds for some $a\perp_{\mathbf{M}}b$. Then R(a',b') holds for all $a'\models tp(a/\mathbf{M})$ and $b'\models tp(b/\mathbf{M})$ with $a'\perp_{\mathbf{M}}b'$.

Proof : Suppose R(a',b') fails for some $a'\models tp(a/\mathbf{M})$ and $b'\models tp(b/\mathbf{M})$ with $a'\perp_{\mathbf{M}}b'$. By **M**-invariancy of R we may assume that a'=a. Let **N** be an $card(\mathbf{M})^+$-saturated and –homogeneous elementary extension of **M**, and let p be a coheir of $tp(b/\mathbf{M})$ to **N**. Le $(b_i : i<\omega)$ and $(b_i' : i<\omega)$ be coheir sequences in p over **M**, with $b=b_0$ and $b'=b_0'$. By Lemma 2.5.6 both sequences are

M-independent and **M**-indiscernible. Since $a \perp_{\mathbf{M}} b'$, by Lemma 2.3.6 we may assume that both sequences are indiscernible over $\mathbf{M} \cup \{a\}$. From this follows that $R(a,b_i)$ holds for all $i<\omega$, and $R(a,b_i')$ fails for all $i<\omega$.

Now let $(c_i : i<\omega)$ be a coheir sequence in p over $\mathbf{M} \cup \{b_i b_i' : i<\omega\}$. Then $(b_i : i<\omega) \wedge (c_i : i<\omega)$ and $(b_i' : i<\omega) \wedge (c_i : i<\omega)$ are coheir sequences in p over **M**, and thus **M**-independent and **M**-indiscernible.

Every infinite increasing subsequence of $(c_i : i<\omega)$ is also a coheir sequence in p over $\mathbf{M} \cup \{b_i b_i' : i<\omega\}$. Consider the set X of formulas expressing that $(x_i : i<\omega)$ is indiscernible over $\mathbf{M} \cup \{a\}$ and $(x_i : i<\omega)$ is a coheir sequence in p over $\mathbf{M} \cup \{b_i b_i' : i<\omega\}$. By Ramsey's Theorem, any finite subset of X is satisfiable by some infinite increasing subset of $(c_i : i<\omega)$, hence X is consistent. Thus, we may assume that $(c_i : i<\omega)$ is in fact indiscernible over $\mathbf{M} \cup \{a\}$.

Now either $R(a,c_i)$ holds for all $i<\omega$, in which case the sequence $(c_i : i<\omega) \wedge (b_i' : i<\omega)$ contradicts Lemma 2.6.7, or $R(a,c_i)$ fails for all $i<\omega$ and $(b_i : i<\omega) \wedge (c_i : i<\omega)$ contradicts Lemma 2.6.7.

q.e.d.

Corollary 2.6.9 : **The Independence Theorem (over a Model)**

Let **M** be a model, $p \in S(\mathbf{M})$, A, B supersets of **M** with $A \perp_{\mathbf{M}} B$, and $p_A \in S(A)$ and $p_B \in S(B)$ non-forking extensions of p. Then $p_A \cup p_B$ does not fork over **M**.

Proof : Let $a \models p_A$ and $a' \models p_B$. Since a and a' both realize p, there is an **M**-automorphism mapping a' to a and B to some B'. By Proposition 2.4.19(ii) there is an $\mathbf{M} \cup \{a\}$-automorphism mapping B' to some B'' with $B'' \perp_{\mathbf{M}a} A$. So a realizes p_A and $p_{B''}$, the isomorph image of p_B.

As p_A is a non-forking extension of p and hence $A \perp_{\mathbf{M}} a$, we get $A \perp_{\mathbf{M}} B''$, by transitivity. Furthermore $A \perp_{B''} a$, since $\mathbf{M} \subseteq B''$. As p_B is a non-forking extension of p and hence $a' \perp_{\mathbf{M}} B$, it follows that $a \perp_{\mathbf{M}} B'$, whence $a \perp_{\mathbf{M}} B''$, since independence is invariant under automorphisms. So we obtain $a \perp_{\mathbf{M}} AB''$ (see 2.4.19(ii)). Therefore $p_A \cup p_{B''}$ does not fork over **M**. But "$p_X \cup p_Y$ does not fork over **M**" is a stable **M**-invariant relation by Lemma 2.6.6, so $p_A \cup p_B$ does not fork over **M**, by Theorem 2.6.8.

q.e.d.

Corollary 2.6.10 : Let **M** be a model, $p \in S(\mathbf{M})$, $(A_i : i \in I)$ an independent sequence over **M**, and for each $i \in I$ consider a non-forking extension p_i of p to A_i. Then $\cup_{i \in I} p_i$ is consistent and does not fork over **M**.

Proof : Since consistency and non-forking are local properties, we only have to check the assertion for finite I. But here it follows by induction on card(I) from the Independence Theorem over a model.

q.e.d.

In simple theories a type over a model may have more than one non-forking extension to a given superset (this is not the case in stable theories, where a type over a model or over an algebraic closed set is stationary, see Chapter 2.11). For types over models or algebraic closed sets one can show the following two Lemmas:

Lemma 2.6.11 : Suppose $p \in S(\mathbf{M})$ has only a bounded number of non-forking extensions (that is, less than κ'=card(**C**)-many). Then p is stationary.

Proof : Suppose not and let A be a superset of **M**, such that there are two distinct non-forking extensions $p_1(x,A)$ and $p_2(x,A)$ over A. Let $(A_i : i<\alpha)$ be a Morley sequence in tp(A/**M**) for some cardinal α. By Corollary 2.6.10 for every $I\subseteq\alpha$ there is a non-forking extension q_I of p extending $(\cup_{i\in I}p_1(x,A_i))\cup(\cup_{j\notin I}p_2(x,A_j))$. Clearly $q_I\neq q_J$ for $I\neq J$. But α has 2^α-many subsets, so p cannot have a bounded number of non-forking extensions.

q.e.d.

Proposition 2.6.12 : Suppose A=acl(A) and $p\in(A)$ has only a bounded number of non-forking extensions. Then p is stationary.

Proof : Let **M** be a model containing A. By the previous Lemma it is sufficient to show that p has a unique non-forking extension to **M**. So suppose p_1 and p_2 are two non-forking extensions to **M**.

Claim : p_1 is definable over A.

Proof of the claim: By Corollary 2.4.22, for every formula $\varphi(x,y)$ there is a φ-definition $d_1\varphi(y)$ over **M**. Now if $d_1\varphi$ were not over A, then by compactness it would have arbitrarily many A-conjugates (Let $b\subseteq\mathbf{M}$ be the tuple consisting of the parameters in $d_1\varphi$ and consider the type tp(b/A). Since $b\notin A=acl(A)$, compactness yields arbitrarily many realisations of this type.). All these A-conjugates would be φ-definitions for distinct non-forking extensions of p. But this contradicts boundedness. q.e.d. claim.

Similarly, p_2 is definable over A. Let $d_2\varphi$ be the φ-definition for p_2, and consider some $b\in\mathbf{M}$. By Lemma 2.4.20 there is a partial type q(x) such that $\models q(a')$ if and only if $\varphi(a',y)$ is in some non-forking extension of tp(b/A) to Aa'.

Now suppose $\varphi(x,b)\in p_1$. If $a_1\models p_1$, then $\models\varphi(a_1,b)$, whence $q(a_1)$, since $a_1\perp_A b$ and $\varphi(a_1,b)\in tp(b/Aa_1)$. If $b\in A\subseteq\mathbf{M}$, then $\varphi(x,b)\in p$, and by the same argument follows that any element realizing p also realizes q. So $q\subseteq p$. In particular, $\models q(a_2)$ for any $a_2\models p_2$. Hence, there is some $b'\models tp(b/A)$ such that $b'\perp_A a_2$, and $\varphi(a_2,y)\in tp(b'/Aa_2)$, which is a non-forking extension of tp(b/A). So $\models\varphi(a_2,b')$, and therefore $\models d_2\varphi(b')$. But this formula is over A, and since b' $\models$tp(b/A) we get by some A-automorphism $\models d_2\varphi(b)$. This means that $\varphi(x,b)\in p_2$. Thus, $p_1\subseteq p_2$, and similarly $p_2\subseteq p_1$, hence $p_1=p_2$, and p is stationary.

q.e.d.

Definition 2.6.13 : Let A be a set of parameters. The group of *strong* automorphisms of **C** over A is the subgroup of Aut(**C**/A) generated by all automorphisms fixing some model **M** which contains A. We denote it by Autf(**C**/A). Sometimes it is also called the Lascar group.

Two tuples a, b have the same *Lascar strong type* over A if they are conjugate by a strong automorphism over A. We write Lstp(a/A)=Lstp(b/A).

So a and b have the same Lascar strong type if there are an $n<\omega$ and tuples $a=a_0, \dots , a_{n-1}$, $a_n=b$, and models $\mathbf{M}_0, \dots, \mathbf{M}_{n-1}$, each of them containing A, such that $tp(a_i/\mathbf{M}_i)=tp(a_{i+1}/\mathbf{M}_{i-1})$, for any $i\leq n$.

It is clear that equality of Lascar strong types is an equivalence relation, and Lstp(a/A) denotes the equivalence class of a. In other words, Lstp(a/A) represents the orbit of a under Autf(**C**/A). Trivially, if A is a model, then Autf(**C**/A)=Aut(**C**/A), so types over models are the same as Lascar strong types.

Lemma 2.6.14 : Suppose $A\subseteq\mathbf{M}$ and tp(a/**M**)=tp(b/**M**). Then there is a model **M**' containing A with $ab\perp_A\mathbf{M}'$ and tp(a/**M**')=tp(b/**M**').

Proof : tp(**M**/A) does not fork over A and can be considered as a partial type over some model $\mathbf{M}_0$, with $A\subseteq\mathbf{M}_0$. By Extension (Theorem 2.4.18), the type as a completion which does not fork over A, hence $\mathbf{M}_0 \perp_A \mathbf{M}$. Let **M**' realize a coheir of tp($\mathbf{M}_0$/**M**) to $\mathbf{M} \cup \{a,b\}$. Then **M**' contains A. Furthermore, by Lemma 2.3.5, $\mathbf{M}' \perp_{\mathbf{M}} ab$, and by transitivity we get $\mathbf{M}' \perp_A \mathbf{M}ab$. Again by Lemma 2.3.5 we obtain tp(**M**'a/**M**)=tp(**M**'b/**M**). This yields the assertion.

q.e.d.

Theorem 2.6.15 : Suppose $a \perp_A b$ and Lstp(a/A)=Lstp(b/A). Then there is a model **M** containing A with $ab \perp_A \mathbf{M}$ and tp(a/**M**)=tp(b/**M**).

Proof : There are $n<\omega$ and a sequence $a=a_0, a_1, \dots, a_n=b$ and models $\mathbf{M}_0, \dots, \mathbf{M}_{n-1}$ containing A, such that $tp(a_i/\mathbf{M}_i)=tp(a_{i+1}/\mathbf{M}_i)$ for all $i<n$. Put $C=\cup_{i<n}\mathbf{M}_i$.

Claim 1 : We may assume $(a_i : i\leq n) \perp_A C$.

Proof of the claim : By Lemma 2.6.14 we may assume that $a_i a_{i+1} \perp_A \mathbf{M}_i$ for all $i<n$. Conjugating $\mathbf{M}_i$ over Aa_ia_{i+1} (see Proposition 2.4.19(ii)), we may also assume inductively that $\mathbf{M}_i \perp_{Aa_ia_{i+1}} a_0 \dots a_n \mathbf{M}_0 \dots \mathbf{M}_{i-1}$ for all $i<n$. Then $\mathbf{M}_i \perp_A a_0 \dots a_n \mathbf{M}_0 \dots \mathbf{M}_{i-1}$ for all $i<n$, by transitivity. Hence, $a_0 \dots a_n \perp_A \mathbf{M}_0 \dots \mathbf{M}_{n-1}$ (see Proposition 2.4.19(iii)). Q.e.d. claim 1.

Claim 2 : We may assume that $(a_i : i\leq n)$ is independent over C.

Proof of the claim : For $0<i<n$ choose inductively realizations a_i' of some non-forking extension of $tp(a_i/C)$ to $Ca_0'a_n'a_1'\dots a_{i-1}'$ (this non-forking extensions exist by Extension (Theorem 2.4.18)). Then $(a_i' : i\leq n)$ is independent over C, and since $tp(a_i'/C)=tp(a_i/C)$ and C contains all $\mathbf{M}_i$, for $i<n$, the new sequence still witnesses the equality of Lascar strong type. q.e.d. claim 2.

Note that from claim 1 and claim 2 now follows that $(a_i : i\leq n)$ is independent over A: Since $a_i \perp_A C$ and $a_i \perp_C (a_j : j<i)$ we get by transitivity $a_i \perp_A (a_j : j<i)$, for all $i\leq n$.

We now use induction on n. If n=1, the assertion is already shown and follows from Lemma 2.6.14. So assume it is true for n-1. Since $(a_i : i\leq n)$ is independent over A and in particular $a_0 \perp_A a_{n-1}$ (see the remark after Definition 2.5.1) and $Lstp(a_0/A)=Lstp(a_{n-1}/A)$, by induction hypothesis there is a model **M** containing A such that tp(a_0/**M**)=tp(a_{n-1}/**M**). This means that we may assume that n=2.

Since $a_0a_1a_2 \perp_A$ **M**$_0$**M**$_1$, we get $a_0a_1a_2 \perp_{\mathbf{M}_0}$ **M**$_1$. Furthermore, independence of $\{a_0,a_1,a_2\}$ over **M**$_0$**M**$_1$ and transitivity imply $a_0 \perp_{\mathbf{M}_0} a_1a_2$. Let p=tp(**M**$_1$/**M**$_0a_1a_2$), and p' the conjugate over **M**$_0a_0$ of tp(**M**$_1$/**M**$_0a_1$), which exists since tp(a_0/**M**$_0$)=tp(a_1/**M**$_0$). Then both p and p' are non-forking extensions of tp(**M**$_1$/**M**$_0$). By the Independence Theorem 2.6.9 over the model **M**$_0$ they have a common realization **M**, such that **M**$\perp_{\mathbf{M}_0} a_0a_1a_2$.

We have tp(a_0**M**)=tp(a_1**M**$_1$) (since **M** $\models$p'$\in$S(**M**$_0a_0$) and p' is the conjugate of tp(**M**$_1$/**M**$_0a_1$)

=tp(a_2**M**$_1$) (by hypothesis)

=tp(a_2**M**) (since **M** $\models$p=tp(**M**$_1$/**M**$_0a_1a_2$)),

hence, tp(a_0/**M**)=tp(a_2/**M**), and the assertion is shown.

q.e.d.

Theorem 2.6.16 : **The Independence Theorem (for Lascar strong types)**

If $B \perp_A C$, tp(b/AB) and tp(c/AC) do not fork over A, and Lstp(b/A)=Lstp(c/A), then there is a $\models$Lstp(b/A)$\cup$tp(b/AB)$\cup$tp(c/AC), with $a \perp_A BC$ (that is, the union of the two non-forking extensions does not fork over A).

Proof : Let **M** be a model containing A with **M**$\perp_A$BCbc (see Proposition 2.4.19(ii)), and let b' and c' realize non-forking extensions of tp(b/**M**B) to **M**BC and tp(c/**M**C) to **M**BCb', respectively. Replacing b by b' and c by c' then preserves the types and Lascar strong types in question. We may thus assume $Bb \perp_A Cc$ (by transitivity).
By Theorem 2.6.15 there is a model **M'** containing A, such that $bc \perp_A$ **M'** and tp(b/**M'**)=tp(c/**M'**). By Proposition 2.4.19(ii) we may assume that **M'**$\perp_{Abc}$BC, whence **M'**$\perp_A$BCbc, by transitivity. But then $B \perp_{\mathbf{M}'} C$ (since A$\subseteq$**M'** and $B \perp_A C$), and tp(b/**M'**B) and tp(c/**M'**C) both do not fork over **M'** (since A$\subseteq$**M'**) and have the same restriction to **M'**. By

the Independence Theorem over **M'** they have a common realization a wit $a\perp_{M'}BC$. So $a\perp_A BC$, as tp(a/**M'**)=tp(b/**M'**) does not fork over A and by transitivity. This proves the theorem.
q.e.d.

Remark 2.6.17 : The Independence Theorem for Lascar strong types is a generalization of the Independence Theorem over a model (Corollary 2.6.9), both due to Kim, Pillay [Kim1]. The second is exactly the same assertion as the first Independence Theorem, substituting a type over a model by a Lascar strong type over some set, whence, it is a considerably stronger result (since Lascar strong type over a model is the same as the type over the model).
Pillay and Kim, tried to show the Independence Theorem for types over algebraic closed sets, but they was not able to do it. However, they discovered that the notion of Lascar strong type is exactly what is needed to generalize the Independence Theorem over a model to arbitrary sets. Later we will define the bounded closure of A, bdd(A), which contains acl(A), and see that Lstp(a/A)=tp(a/bdd(A)). Therefore Lstp(a/A) $\models$tp(a/acl(A)). See the discussion at the beginning of chapter 2.10.

Recently, Casanovas was able to prove the Independence Theorem for Lascar strong types by a shorter proof, using new results about indiscernible sequence in simple theories [Cas5].

2.7 Simplicity and Independence

Theorem 2.7.1 : Suppose in a complete theory T there is an abstract independence relation $\perp^0$, invariant under automorphisms, such that the following hold

1. SYMMETRY $a\perp^0_A b$ if and only if $b\perp^0_A a$.
2. TRANSITIVITY $a\perp^0_A BC$ if and only if $a\perp^0_A B$ and $a\perp^0_{AB} C$.
3. EXTENSION For any a, A, B there is a' $\models$tp(a/A) with $a'\perp^0_A B$.
4. LOCAL CHARACTER For any a, A there is A'$\subseteq$A with card(A')$\leq$card(T) and $a\perp^0_{A'} A$.

5. FINITE CHARACTER $a\perp^0_A B$ if and only if for all finite $b\in B$ we have $a\perp^0_A b$.
6. INDEPENDENCE THEOREM OVER A MODEL If $a\perp^0_{\mathbf{M}} b$ for some model $\mathbf{M}$, $x\perp^0_{\mathbf{M}} a$, $y\perp^0_{\mathbf{M}} b$ and $tp(x/\mathbf{M})=tp(y/\mathbf{M})$, then there is $z\models tp(x/\mathbf{M}a)\cup tp(y/\mathbf{M}b)$ with $z\perp^0_{\mathbf{M}} ab$.

Then T is simple and $\perp^0$ is non-forking independence.

Proof : We shall first show that if $a\perp^0_A b$ and $(b_i : i<\omega)$ is an A-indiscernible sequence in $tp(b/A)$, then there is $a'\perp^0_A(b_i : i<\omega)$ such that $tp(a'b_i/A)=tp(ab/A)$ for all $i<\omega$. (This implies that $\cup_{i<\omega}p_i(x,Ab_i)$, for $p(x,Ab)=tp(a/Ab)$, is consistent, and hence, $tp(a/Ab)$ does not divide over A.)
First, by compactness, we extend $(b_i : i<\omega)$ to an A-indiscernible sequence $(b_i' : i\leq card(T)^+)$. For all $i\leq card(T)^+$ we then successively find models $\mathbf{M}_i$ of cardinality card(T), such that $A\cup(\cup_{j<i}\mathbf{M}_j b_j')\subseteq\mathbf{M}_i$, and $(b_j' : i\leq j\leq card(T)^+)$ is indiscernible over $\mathbf{M}_i$, in the following way: Suppose that $\mathbf{M}_j$ have been found for $j<i$. Let $\mathbf{M}_i'$ be any model of size card(T) containing $A\cup(\cup_{j<i}\mathbf{M}_j b_j')$. By Ramsey's Theorem and compactness we obtain an $\mathbf{M}_i'$-indiscernible sequence $(b_j'' : i\leq j\leq card(T)^+)$ whose type over $\cup_{j<i}\mathbf{M}_j$ is the same as that of $(b_j' : i\leq j\leq card(T)^+)$. So we choose $\mathbf{M}_i$ to be the image of $\mathbf{M}_i'$ under a $\cup_{j<i}\mathbf{M}_j$-isomorphismus mapping $(b_j'' : i\leq j\leq card(T)^+)$ to $(b_j' : i\leq j\leq card(T)^+)$.
Put $k=card(T)^+$. By local character and Transitivity of $\perp^0$ there is some $i<k$ such that $b_k'\perp^0_{\mathbf{M}_i}\cup_{j<k}\mathbf{M}_j$. So by transitivity again, $b_k'\perp^0_{\mathbf{M}_i}(b_j' : i\leq j<k)$. By Transitivity and $\mathbf{M}_i$-indiscernibility we obtain: (*) $b_j'\perp^0_{\mathbf{M}_i}(b_k' : k<j)$, for all $i\leq j<k$.
Clearly, we may assume $b_j=b_{i+j}'$, for all $j<\omega$, and $b=b_0$, whence $b_0=b_i'$. By extension, there is $a'\models tp(a/Ab)$ with $a'\perp^0_{Ab}\mathbf{M}_i$, hence $a'\perp^0_A\mathbf{M}_i b_0$, by Transitivity, and by Transivity again we get $a'\perp^0_{\mathbf{M}_i} b_0$. Since (by our assumption $b_j=b_{i+j}'$) $(b_j : j<\omega)$ is $\perp^0$-independent over $\mathbf{M}_i$ (by (*)), repeated applications of the Independence Theorem over $\mathbf{M}_i$ (in the same way like Corollary 2.6.10 for $\perp$-independence) yield some a'' with $a''\perp^0_{\mathbf{M}_i}(b_j : j<\omega)$ such that $tp(a''b_j/A)=tp(a'b_0/A)$, for all $j<\omega$. This clearly implies $tp(a''b_j/A)=tp(ab/A)$, for all $j<\omega$. And by transitivity, $a''\perp^0_A(b_j : j<\omega)$ (since $a''\perp^0_A\mathbf{M}_i$ and $a''\perp^0_{\mathbf{M}_i}(b_j : j<\omega)$).

Next, we shall show that $a \perp^0_A B$ if and only if tp(a/AB) does not divide over A. This will imply symmetry (and all the other properties) for non-dividing, and thus simplicity of T. Clearly, if $a \perp^0_A B$, then (using Finite Character) we have just seen that tp(a/Ab) does not divide over A for all finite $b \in B$, so tp(a/AB) does not divide over A.

Conversely, assume that tp(a/Ab) does not divide over A. By Extension, there is some $b_0 \models tp(b/A)$ such that $b_0 \perp^0_A A$. Now, again by Extension, there is some $b_1 \models tp(b_0/A)$ such that $b_1 \perp^0_A b_0$. So, inductively we find a sequence $(b_i : i \leq \alpha)$, for big α, such that $b_i \models tp(b/A)$ and $b_i \perp^0_A (b_j : j<i)$, for all $i \leq \alpha$. By Proposition 2.2.5 and compactess, we may assume that $(b_i : i \leq \alpha)$ is A-indiscernible. As $p(x,Ab)=tp(a/Ab)$ does not divide over A, $\cup_{i \leq \alpha} p(x,Ab_i)$ is consistent. In other words, there is some a' such that $tp(a'b_i/A)=tp(ab/A)$, for all $i \leq \alpha$. By Local Character for $\perp^0$ there is some $i<\alpha$ such that $a' \perp^0_{A \cup (bj : j<i)} b_\alpha$.As $b_\alpha \perp^0_A (b_j : j<i)$, we get by Transitivity $b_\alpha \perp^0_A \{a', b_j : j<i\}$, whence $b_\alpha \perp^0_A a'$, by Transitivity again. By invariance of $\perp^0$ under automorphisms, $a \perp^0_A b$.

q.e.d.

2.8 Bounded Equivalence Relations

In this section, T is not necessarily simple. We shall study equivalence relations in an arbitrary theory. The results of this chapter are useful in their own sake but turn out to be also very important for the study of Lascar strong types and strong types. Most of this material here presented was developed in detail during a seminar at the University of Barcelona under the participation of the author. The results 2.8.34 – 39 are due to the author. They imply the results 2.8.43 and 2.8.44.

Recall that a relation R is A-invariant, if it is invariant under A-automorphisms (see Definition 2.6.4).

A sequence of the form $(a_i : i \in I)$ we shall call I-sequence.

Lemma 2.8.1 : R is A-invariant if and only if there is a family of partial types $\{p_i(x) : i\in I\}$ over A such that

$$R(a) \text{ if and only if } \models \vee_{i\in I} p_i(a).$$

Proof : We put $\{p_i(x) : i\in I\}=\{tp(a/A) : R(a)\}$.

Definition 2.8.2 : A relation R is *type-definable over A* if there is a partial type p(x) over A such that for any a

$$R(a) \text{ if and only if } p(a),$$

that is, R=p(**C**). If p is finite (or, equivalently, if p consists only of one formula) then we say that R is *definable over A*.

Remark : It is obvious, that definability of R implies type-definability of R, and this implies invariance of R.
Since Aut(**C**/A) in the theory T coincide with Aut(**C**/∅)=Aut(**C**) in the theory T(A), it is clear that a relation R is A-invariant in T if and only if R is invariant in T(A). It is also clear that a relation R is type-definable over A in the theory T if and only if it is type-definable over ∅ in the theory T(A).

Lemma 2.8.3 : If R is A-invariant and type-definable, then it is type-definable over A.

Proof : If R is given by the partial type p(x,B) for some parameters B, then it is also given by $\exists Y[p(x,Y)\wedge Y \models tp(B/A)]$. Obviously, this can be expressed by an infinite set of formulas in first order logic, hence by a partial type over A.
q.e.d.

Definition 2.8.4 : Let R be a binary relation between I-sequences. R is *finite* if there is some $n<\omega$ such that there is no sequence $(a_i : i<n)$ with $\neg R(a_i,a_j)$, for any $i<j<n$. R is *bounded* if for some cardinal κ there is no sequence $(a_i : i<\kappa)$ with $\neg R(a_i,a_j)$, for any $i<j<\kappa$. n, κ are called *bounds* of R, respectively.

If R is an equivalence relation, then R is finite if it has only a finite number of equivalence classes, and it is bounded if it has only a bounded number of equivalence classes, that is, the number of classes is a cardinal number.
Note that if R is definable and bounded then, by compactness, R must be finite.

Lemma 2.8.5 : A bounded intersection of bounded relations is a bounded relation.

Proof : Let card(I)=λ, and let (R_i : $i\in I$) be a family of bounded relations. For every $i\in I$ let κ_i be a bound of R_i, and let $\kappa=\sup\{\kappa_i : i\in I\}$. Then $\kappa\geq\lambda$.
Suppose that $R=\cap_{i\in I}R_i$ were not bounded. Then, in particular, R is not bounded by $(2^\kappa)^+$. So there exist a sequence S of cardinality $(2^\kappa)^+$ of tuples which are not related by R. Since $\neg R(x,y)$ implies $\neg R_i(x,y)$, for some $i\in I$, we can assign to every pair of tuples of this sequence an $i\in I$ with the property that it is not related by R_i. Painting the pairs of tuples in this way in λ-many colors (note that $\lambda\leq\kappa$), by the Erdös-Rado-Theorem (Theorem 2.2.4) we get a subsequence $S'\subseteq S$ of cardinality κ^+ of tuples which are not related by R_i, for some fixed $i\in I$. But the bound of R_i is $\kappa_i<\kappa^+$, a contradiction. So R must be bounded (by $(2^\kappa)^+$).
q.e.d.

Lemma 2.8.6 : If E is a bounded A-invariant equivalence relation, and (a_i : $i\in I$) is an infinite A-indiscernible sequence, then $E(a_i,a_j)$, for all $i\neq j$ in I.

Proof : By compactness, for any cardinal κ we can find an A-indiscernible sequence (b_i : $i<\kappa$) with $tp(a_i,a_j/A)=tp(b_0,b_1/A)$, for any $i<j$ in I. So if $\neg E(a_i,a_j)$ holds, then $\neg E(b_k,b_l)$, for all $k<l<\kappa$, by A-invariance of E and A-indiscernibility. Then E has κ many classes, a contradiction.
q.e.d.

The following Lemma supplies a bound for the number of equivalence classes of an invariant, bounded equivalence relation. However, we will see below (Remark 2.8.22) that there exists a better bound.

Lemma 2.8.7 : Let E be an invariant equivalence relation between I-sequences, and let $\kappa=2^{card(I)+card(T)}$. If E has more than 2^κ classes, then E is not bounded.

Proof : Suppose that E has $\geq(2^\kappa)^+$ classes. There are at most κ many types tp(a,b) of I-sequences a and b. So by Erdös-Rado (Theorem 2.2.4), there is an infinite set $(a_i : i<\omega)$ representing infinitely many classes of E such that $tp(a_i,a_j)=tp(a_l,a_k)$, for i<j and l<k. By compactness, for any cardinal λ there is a sequence $(b_i : i<\lambda)$ such that $tp(b_i,b_j)=tp(a_l,a_k)$, for $i<j<\lambda$ and $l<k<\omega$. Since E is invariant we get $\neg E(b_i,b_j)$, for $i<j<\lambda$. So E has at least λ many classes and is not bounded.
q.e.d.

In the following we will define three equivalence relations between I-sequences in an arbitrary complete theory T. We shall omit the reference to I in the notation assuming that it is clear from the context.

Definition 2.8.8 :

1. $E_L^A(a,b)$ if and only if E(a,b) for any bounded and A-invariant equivalence relation E
2. $E_{KP}^A(a,b)$ if and only if E(a,b) for any bounded and over A type-definable equivalence relation E
3. $E_{Sh}^A(a,b)$ if and only if E(a,b) for any finite and over A definable equivalence relation E

Remark 2.8.9 : It is clear that $E_L^A(a,b)$ holds in the theory T if and only if $E_L^\varnothing(a,b)$ holds in the theory T(A), see the remark following Definition 2.8.2. Similarily for E_{KP}^A and E_{Sh}^A. So, to simplify matters, we assume in the following that $A=\varnothing$ and write E_L, E_{KP}, E_{Sh}.
The letters "L", "KP", "Sh" refer to the names of Lascar, Kim/Pillay, and Shelah, respectively.

Proposition 2.8.10 :

1. $E_L\subseteq E_{KP}\subseteq E_{Sh}$.

2. E_L is bounded and invariant, and refines every bounded and invariant equivalence relation.
3. E_{KP} is bounded and type-definable over $\emptyset$, and refines every bounded and over $\emptyset$ type-definable equivalence relation.
4. E_{Sh} is bounded and type-definable over $\emptyset$, and refines every finite and over $\emptyset$ definable equivalence relation.

Proof : 1. is obvious by the definitions.

3: E_{KP} is bounded by Lemma 2.8.5 and clearly type-definable over $\emptyset$, since it is definable by the union of partial types over $\emptyset$.

4: E_{Sh} is definable by the set of all formulas which define a finite and over $\emptyset$ definable equivalence relation. Suppose it is not bounded. Then, by compactness, we can find a sequence of cardinality $(2^{\kappa})^{+}$, for κ=card(T), of tuples which are not related by E_{Sh}. Since there are at most κ many formulas defining a finite equivalence relation, and $\neg E_{Sh}(x,y)$ implies $\neg E(x,y)$, for some definable, finite equivalence relation, by Erdös-Rado some of these formulas has infinite many equivalence classes, a contradiction.

2: We shall prove that E_L is bounded. It is a relation between I-sequences. Let $\lambda=2^{\text{card(I)+card(T)}}$, $\kappa=2^{\lambda}$, and let **M** be a κ^{+}-saturated model.

Claim 1: **M** contains representative elements of all equivalence classes of all bounded, invariant equivalence relations.

Proof of claim 1: Consider a bounded and invariant equivalence relation E, and a maximal sequence $(a_i : i<\alpha)$ such that $\neg E(a_i,a_j)$, for $i<j<\alpha$. By Lemma 2.8.7: $\alpha\leq\kappa$. Consider the type $p=tp((a_i : i<\alpha))$. Since **M** is κ^{+}-saturated it contains a realization of p, in other words, **M** contains a representative for every equivalence class of E.

q.e.d. claim 1.

Claim 2: tp(a/**M**)=tp(b/**M**) implies $E_L(a,b)$.

Proof of claim 2: Let E be any bounded and invariant equivalence relation, and let f be an **M**-automorphism mapping a to b. By claim 1, there is $a'\in$**M** such that E(a,a'). Let $[a]_E$ be the equivalence class of a. Since E is invariant, we get $f([a]_E)=[b]_E$. On the other hand: $f([a]_E)=f([a']_E)=[f(a')]_E=[a']_E=[a]_E$, since f fix **M**. Hence, $[a]_E=[b]_E$, and E(a,b).

q.e.d. claim 2.

So E_L is bounded, since the number of types of I-sequences over **M** is bounded.

q.e.d.

Definition 2.8.11 : A formula $\varphi(x,y)$ is finite, if there is no infinite sequence $(a_i : i<\omega)$ with $\models\neg\varphi(a_i,a_j)$, for all $i<j<\omega$. A formula $\varphi(x,y)$ is thick, if it is finite, reflexive and symmetric.

Definition 2.8.12 : Let $\varphi(x,y)$ and $\psi(x,y)$ be formulas with the property that x and y are sequences of variables of the same length. The *product* of φ and ψ is the formula

$$(\varphi\circ\psi)(x,y)=\exists z(\varphi(x,z)\wedge\psi(z,y)).$$

Let $p_1(x,y)$ and $p_2(x,y)$ be two partial types such that the sequences x and y have the same length. The product of p_1 and p_2 is the type

$$(p_1\circ p_2)(x,y)=\exists z(p_1(x,z)\wedge p_2(z,y)).$$

(This means that $\models(p_1\circ p_2)(a,b)$ if and only if there exists some sequence c such that $\models p_1(a,c)$ and $\models p_2(c,b)$. So, by compactness, the product of p_1 and p_2 is in fact the type $\{\varphi_1(x,y)\circ\varphi_2(x,y) : \varphi_1\in p_1, \varphi_2\in p_2\}$.)

Iterating the product, we get the definition of the *n^{th} power* $p^n(x,y)$ of some type $p(x,y)$, $n<\omega$.

Note that compactness implies that $p^n(x,y)$ is axiomatizable by $\{\varphi^n(x,y) : \varphi\in p\}$.

If p_1 defines the relation R_1 and p_2 defines the relation R_2, then it is clear that $p_1\circ p_2$ defines the product $R_1\circ R_2$ of these two relations.

Proposition 2.8.13 :

1. If $\varphi(x,y)$ is finite and $\varphi(x,y)\vdash\psi(x,y)$, then ψ is finite.
2. Let $\varphi(x,y)$ be thick and $\psi(x,y)$ symmetric and suppose that $\varphi\vdash\psi$. Then ψ is thick.
3. If $\varphi(x,y)$ and $\psi(x,y)$ are finite (thick), then $\varphi(x,y)\wedge\psi(x,y)$ is finite (thick).
4. If $\varphi(x,y)$ and $\psi(x,y)$ are finite (thick), then $\varphi(x,y)\vee\psi(x,y)$ is finite (thick).
5. If $\varphi(x,y)$ is thick, then $\varphi^n(x,y)$ is thick.
6. If $\varphi(x,y)$ and $\psi(x,y)$ are thick, then $\varphi(x,y)\circ\psi(x,y)$ is thick.

Proof :

3: Suppose $\varphi(x,y)\wedge\psi(x,y)$ is not finite. Then there is a sequence $(a_i : i<\omega)$ such that $\models\neg\varphi(a_i,a_j)\vee\neg\psi(a_i,a_j)$, for all $i<j<\omega$. So by Ramsey's Theorem, there is an infinite subset $I\subseteq\omega$, such that either $\models\neg\varphi(a_i,a_j)$ for all $i<j<I$, or $\models\neg\psi(a_i,a_j)$ for all $i<j<I$. Hence, either φ is not finite or ψ is not finite.

6: Suppose $\varphi(x,y)\circ\psi(x,y)$ is not finite. Since $\neg\exists z.\varphi(x,z)\wedge\psi(z,y)$ implies $\neg(\varphi(x,y)\wedge\psi(y,y))$ implies $\neg\varphi(x,y)$, we get that $\varphi(x,y)$ is not finite.

The rest of the assertions is obvious.

q.e.d.

Definition 2.8.14 : $nc_A(x,y)$ is the set of all thick formulas with parameters in A.

Remark 2.8.15 : If $A=\emptyset$, we write $nc(x,y)$. Since $\models nc_A(a,b)$ in T if and only if $\models nc(a,b)$ in T(A), and most of our assertions do not depend of the constants contained in the theory in which we are working, we may omit the set A in the notation.

$nc(x,y)$ is a partial type, closed under conjunction, disjunction and product. If $\varphi(x,y)$ is symmetric and $nc(x,y)\vdash\varphi(x,y)$, then $\varphi(x,y)\in nc(x,y)$. This follows by the previous proposition.

Proposition 2.8.16 : The following conditions are equivalent for $a\neq b$:

1. $\models nc(a,b)$.
2. There is an infinite indiscernible sequence $(a_i : i<\omega)$ with $a=a_0$ and $b=a_1$.
3. There is an infinite indiscernible sequence $(a_i : i<\omega)$ and there are some $i\neq j$ such that $a=a_i$ and $b=b_j$.

Proof : Suppose $\models nc(a,b)$. In order to prove 2. it is sufficient to show that there is an infinite sequence $(c_i : i<\omega)$ such that $tp(c_i,c_j)=tp(a,b)$ for any $i<j<\omega$: Then we can apply compactness to extend this sequence to a sequence of arbitrary length with the same properties and use Proposition 2.2.5 to obtain the desired indiscernible sequence.

So let $p(x,y)=tp(a,b)$. We have to show the consistency of the set $\{x_i\neq x_j : i<j<\omega\}\cup(\cup_{i<j<\omega}p(x_i,x_j))$. By compactness, it is sufficient to prove that for every formula

$\varphi(x,y)\in p(x,y)$ there is an infinite sequence $(a_i : i<\omega)$ such that $\models\varphi(a_i,a_j)$ for all $i<j<\omega$. Suppose there is some formula $\varphi\in p$ and no infinite sequence with this property. Then $\neg\varphi(x,y)$ is finite, whence $x=y\vee(\neg\varphi(x,y)\wedge\neg\varphi(y,x))$ is thick and must be in nc(a,b). But this means that $\models\neg\varphi(a,b)$, a contradiction to our assumption $\varphi\in p=tp(a,b)$.

It is clear that 2. implies 3.

3.→1.: Suppose there is some thick formula φ with $\models\neg\varphi(a,b)$. By indiscernibility, we get $\models\neg\varphi(a_i,a_j)$ for all $i<j<\omega$. This contradicts the hypothesis that φ is finite. Hence, $\models\varphi(a,b)$ for all thick formulas φ.

q.e.d.

Observation 2.8.17 :

1. The realizations of the partial type $nc^n(x,y)$ are exactly the pairs (a,b) which are connected by n infinite indiscernible sequences. That is, there are $a_1,\ldots,a_{n+1}$ and indiscernible sequences $I_1,\ldots,I_n$ such that $a=a_1$, $b=a_{n+1}$, and for every $i\leq n$, $a_i, a_{i+1}\in I_i$. (This partial type is axiomatizable by $\{\varphi^n(x,y) : \varphi(x,y)\in nc(x,y)\}$.)
2. $nc(x,y)\supseteq nc^2(x,y)\supseteq nc^3(x,y)\supseteq\ldots$.
3. The transitive closure of the relation defined by nc(x,y) is equivalent to the infinite disjunction $\vee_n nc^n(x,y)$.

Proof : 1. follows immediately from Proposition 2.8.16.

In order to prove 2. we use induction on n. For n=2 the assertion $nc^{n-1}(x,y)\supseteq nc^n(x,y)$ follows from Proposition 2.8.13.6. Now suppose it holds $nc^{n-1}(x,y)\supseteq nc^n(x,y)$ for $n>1$. Let $\varphi(x,y)\in nc^{n+1}(x,y)$. Then we may assume that $\varphi=\psi^{n+1}(x,y)=\psi(x,y)\circ\psi^n(x,y)$, for some $\psi\in nc(x,y)$. By induction hypothesis, $\psi^n\in nc^{n-1}(x,y)$. Hence, there is some $\psi'\in nc(x,y)$ such that $\psi^n=\psi'^{n-1}$, whence $\varphi\in nc^n(x,y)$. (Note that - for any $n<\omega$ - $nc^n(x,y)$ can be considered as the set $\{\varphi_1\circ\ldots\circ\varphi_n : \varphi_i\in nc(x,y) \text{ for } 1\leq i\leq n\}$ and also as the set $\{\varphi^n : \varphi\in nc(x,y)\}$.)

3. is clear, since $\vee_n nc^n(x,y)$ says that there is some $n<\omega$ such that $nc^n(x,y)$.

q.e.d.

Proposition 2.8.18 : $E_L^A(a,b)$ if and only if $\models\vee_n nc_A^n(a,b)$.

(Remark: $\models\vee_n nc_A^n(x,y)$ refers to the infinite disjunction: $\models nc_A(x,y)$ or $\models nc_A^2(x,y)$ or … or $\models nc_A^m(x,y)$ or … .)

Proof : to simplify matters, we assume that $A=\varnothing$.

The relation defined by $\vee_n nc^n(x,y)$ is invariant. By Lemma 2.8.5, we know that nc(x,y) is bounded. Since (by Observation 2.8.17.2) $nc(x,y)\vdash nc^n(x,y)$, for every $n<\omega$, it follows that the relation defined by $\vee_n nc^n(x,y)$ is bounded too. Hence, it extends E_L (Proposition 2.8.10) (note that it is an equivalence relation, by definition). For the other direction we use the transitivity of E_L. So it is sufficient to show that $\models nc(a,b)$ implies $E_L(a,b)$. If a=b, then the assertion is trivial. Otherwise a and b are in an infinite indiscernible sequence I, by Proposition 2.8.5. By compactness, we can assume that I is of arbitrary cardinality. If $\neg E_L(a,b)$, then, by indiscernibility and invariance, $\neg E_L(a',b')$ for all a', b' ordered in the same way in I. But this contradicts boundedness of E_L. Hence, $E_L(a,b)$ holds.

q.e.d.

Lemma 2.8.19 :

1. If $\models nc(a,b)$, then there is some model **M** such that tp(a/**M**)=tp(b/**M**).
2. If there is some model **M** with tp(a/**M**)=tp(b/**M**), then $\models nc^2(a,b)$.

Proof : 1.: We may assume that $a\neq b$, so they start some indiscernible sequence I, by 2.8.5. Fix some model **M**. By the remark following Proposition 2.2.5, there is some model **M**', an automorph image of **M**, such that I is **M'**-indiscernible. Hence, tp(a/**M'**)=tp(b/**M'**).

2.: Suppose tp(a/**M**)=tp(b/**M**). By compactness, it is sufficient to see that for every thick formula $\varphi(x,y)$ there is some c such that $\models\varphi(a,c)\wedge\varphi(c,b)$. So let φ be a thick formula and choose $n<\omega$ maximal such that there is some sequence $(a_i : i<n)$ with $\models\neg\varphi(a_i,a_j)$, for all $i<j<n$. Since the sequence is finite, there is a formula expressing the existence of such sequence with this property. It follows that there is such sequence inside the model **M** (recall that every model of the theory is an elementary substructure of the monster model **C**), so we may assume that $(a_i : i<n)$ is in **M**. By maximality of n, there must be some $i<n$ with $\models\varphi(a,a_i)$. Since tp(a/**M**)=tp(b/**M**) we get $\models\varphi(b,a_i)$, and by symmetry $\models\varphi(a_i,b)$.

q.e.d.

Example [Cas4]: We consider the theory T with an equivalence relation E with infinitely many classes, every class having exactly two elements. Let a, b be two different elements in the same class outside of a model **M**. Then for all m∈**M**: $\models\neg E(a,m)$ and $\models\neg E(b,m)$. Hence, tp(a/**M**)=tp(b/**M**), but a and b are not in some infinite indiscernible sequence, since every equivalence class contains only two elements. So, by Proposition 2.8.16, it does not hold that $\models nc(a,b)$.

Proposition 2.8.20 : $E_L^A(a,b)$ if and only if Lstp(a/A)=Lstp(b/A).

Proof : To simplify matters, we assume that A=∅.

$E_L(a,b)$ → there is some $n<\omega$ such that $nc^n(a,b)$ (by Proposition 2.8.18) → there are models $\mathbf{M}_0,\ldots,\mathbf{M}_{n-1}$ and sequences $a_0,\ldots,a_n$, with $a_0=a$, $a_n=b$, and $tp(a_i/\mathbf{M}_i)=tp(a_{i+1}/\mathbf{M}_i)$, for all i<n (by Lemma 2.8.19.1) → Lstp(a)=Lstp(b), by definition of Lascar strong type.

For the other direction suppose that Lstp(a)=Lstp(b). Then there is some $n<\omega$ and some sequence of elements and models like in the first part of the proof. Lemma 2.8.19.2 then implies that there is some $m<\omega$ such that $nc^m(a,b)$, this means that $\vee_m nc^m(a,b)$, whence $E_L(a,b)$ (by 2.8.18).

q.e.d.

Proposition 2.8.21 : Equality of Lascar strong type over A is the finest bounded A-invariant equivalence relation.

Proof : The Proposition follows immediately from Proposition 2.8.10 (and Remark 2.8.9).

Remark 2.8.22 : We would like to give a proof of Proposition 2.8.21 which does not depend on our previous studies about bounded equivalence relations. This proof also provides a better bound of the number of classes of an A-invariant bounded equivalence relation than Lemma 2.8.7.

Proof of 2.8.21 : Equality of Lascar strong types is clearly an A-invariant equivalence relation. Let **M** be a model containing A with card(**M**)=card(T)+card(A). Then two tuples which have the same type over **M** have the same Lascar strong type over A. It follows that there are at most $2^{\mathrm{card(T)+card(A)}}$ Lascar strong types over A, and the relation is bounded.
Now let E be an A-invariant bounded equivalence relation, and consider two tuples a and b with the same Lascar strong type over A. By transitivity of E it is enough to show that if a and b have the same type over some model **M** containing A, then E(a,b) holds. Let **N** be an card(**M**)$^+$-saturated elementary extension of **M** and p a coheir of tp(a/**M**) over **N**. Let $(c_i : i<\omega)\subseteq$**N** be a coheir sequence in p over **M**$\cup\{a,b\}$. Then both $a\wedge(c_i : i<\omega)$ and $b\wedge(c_i : i<\omega)$ are coheir sequences over **M** and therefore **M**-indiscernible. By Lemma 2.8.6 we have $E(a,c_0)$ and $E(b,c_0)$, whence E(a,b).
q.e.d.

Corollary 2.8.23 : An A-invariant bounded equivalence relation has at most $2^{\mathrm{card(T)+card(A)}}$ many classes.

Proof : Follows from the proof of Propositon 2.8.21 given in the previous Remark.
q.e.d.

Definition 2.8.24 : Let $\alpha\leq\omega$ be an ordinal. A formula $\varphi(x,y)$ is *α-thick* if there is a sequence $(\varphi_i(x,y) : i<\alpha)$ of thick formulas such that $\varphi_0=\varphi$, and for every $0<i<\alpha$, $\varphi_i^2(x,y)\vdash\varphi_{i-1}(x,y)$.
(Note that a thick formula is 1-thick, by definition.)

Observation 2.8.25 :

1. Let $\varphi(x,y)$ be thick. If φ is transitive, then φ is ω-thick.
2. If $\varphi(x,y)$ is α-thick, witnessed by the sequence $(\varphi_i : i<\alpha)$, then for all $i<j<\alpha$: $\varphi_j^k\vdash\varphi_i$, where $k=2^{(j-i)}$. In particular, for all $j<\alpha$ holds $\varphi_j^k\vdash\varphi$, where $k=2^j$.
3. Suppose that $\varphi(x,y)$ is α-thick, witnessed by the sequence $(\varphi_i : i<\alpha)$. If there are $i<j<\alpha$ such that φ_i and φ_j are equivalent ($\models\varphi_i(x,y)\leftrightarrow\varphi_j(x,y)$), then φ_i and φ_j are transitive.
4. If φ is n-thick for every $n<\omega$, then φ is not necessarily ω-thick.

5. If φ is α-thick, then for every $n<\alpha$ there is some thick formula ψ such that $\psi^n \vdash \varphi$.
6. Let φ be thick and $k=2^n$, for some $n<\omega$. Then φ^k is n+1 thick.

Proof : 1.: Consider the sequence $(\varphi_i : i<\omega)$ with $\varphi_i=\varphi$, for all $i<\omega$. By transitivity of φ, this sequence witnesses that φ is ω-thick.

2.: We show the assertion by induction on (j-i). (Note that for any formulas $\psi_1(x,y)$ and $\psi_2(x,y)$, $\psi_1 \vdash \psi_2$ implies $\psi_1^m \vdash \psi_2^m$, for $m<\omega$.) For j-i=1 it is clearly true, since $\varphi_i^2 \vdash \varphi_{i-1}$ holds for all $i<\alpha$. Now suppose that $\varphi_j^k \vdash \varphi_i$ holds for $k=2^{(j-i)}$ and $(j-i)\geq 1$. Since $\varphi_{j+1}^2 \vdash \varphi_j$, it follows that $\varphi_{j+1}^{2k} \vdash \varphi_i$, where $2k=2^{(j-i)+1}$. This shows the assertion for (j-i)+1.

3.: If $\varphi_i(x,y)$ and $\varphi_j(x,y)$ are equivalent, then they n^{th}-powers are equivalent too, for every $n<\omega$. By 2., $\varphi_j^k \vdash \varphi_i$, where $k=2^{(j-i)}$. By equivalence we may assume that $\varphi_i=\varphi_j$. So, we have $\varphi_j^k \vdash \varphi_j$, and $\varphi_i^k \vdash \varphi_i$. Since $k\geq 2$, φ_i and φ_j are transitive.

4.: Note that the sequences witnessing that φ is n-thick can be all different for different n, and must not be related by inclusion. So we can't use compactness to prove that φ is ω-thick.

5.: Suppose $n<\alpha$. By 2., $\varphi_n^k \vdash \varphi$, where $k=2^n$. $\varphi_n^k \in nc^k(x,y) \subseteq nc^n(x,y)$, by Observation 2.8.17. So there is some $\psi \in nc(x,y)$ such that $\varphi_n^k=\psi^n$.

6.: The sequence $(\varphi^{k/l} : l=2^m, 0\leq m\leq n)$ witnesses that φ^k is n+1 thick.

q.e.d.

Proposition 2.8.26 : Suppose that $\varphi(x,y)$ is ω-thick, witnessed by the sequence $(\varphi_i : i<\omega)$. Then the partial type $p(x,y)=\{\varphi_i : i<\omega\}$ type-defines a bounded equivalence relation.

Proof : We need only to show the transitivity of p(x,y). So suppose that $\models p(a,c)$ and $\models p(c,b)$. Hence, $\models \varphi_i(a,c)$ and $\models \varphi_i(c,b)$ for all $i<\omega$. That means, $\models \varphi_i^2(a,b)$ for all $i<\omega$. Since $\varphi_i^2 \vdash \varphi_{i-1}$ for all $0<i<\omega$, it follows that $\models \varphi_i(a,b)$ for all $i<\omega$. So we obtain p(a,b), and p is transitive. By Lemma 2.8.5, p is a bounded relation.

q.e.d.

Proposition 2.8.27 : E_{KP} is type-definable by the partial type consisting of all ω-thick formulas.

Proof : By Proposition 2.8.10, we may fix some partial type q(x,y) which type-defines E_{KP}. The formulas in q must be finite (since q is bounded), so we may assume that q(x,y) consists of thick formulas and is closed under conjunctions. We shall see that these formulas are in fact ω-thick. Let φ_0 be such a formula. Since $q(x,y)\cup q(y,z)\vdash\varphi_0(x,z)$, by compactness there is $\varphi_1\in q$ such that $\varphi_1^2\vdash\varphi_0$. Iterating this argument ω times we obtain a sequence of thick formulas which witnesses that φ_0 is ω-thick.
q.e.d.

Definition 2.8.28 : Let R be a relation between sequences. We define *cl(R)* as the least relation which is type-definable and contains R.

Observation 2.8.29 : cl(R) is type-definable by the set $p(x,y)=\{\varphi(x,y)$: if R(a,b) holds, then $\models\varphi(a,b)\}$.

Proof : Since cl(R) is the intersection of all type-definable relations containing R, it is definable by the union of all partial types defining such a relation. That is, cl(R) is definable by the union of all partial types q(x,y) with the property: If R(a,b), then $\models q(a,b)$, for all a, b. This proves the assertion.
q.e.d.

Proposition 2.8.30 : For φ a symmetric formula, the following conditions are equivalent:

1. $E_L(x,y)\vdash\varphi(x,y)$ (that means: every pair a, b satisfying E_L also satisfies φ; note that E_L is not a set of formulas)
2. $\vee_n nc^n(x,y)\vdash\varphi(x,y)$ (see the remark in 1.)
3. $nc^n(x,y)\vdash\varphi(x,y)$ for every n.
4. For every n<ω there is $\psi\in nc(x,y)$ such that $\psi^n(x,y)\vdash\varphi(x,y)$.
5. φ is n-thick for every n<ω.

Proof : The equivalence of 1. and 2. is clear by Proposition 2.8.18.

2.→3.: Let $n<\omega$ and suppose $nc^n(a,b)$. Then $\vee_n nc(a,b)$, and $\models\varphi(a,b)$, by hypothesis. So for all $n<\omega$ holds $nc^n(x,y)\vdash\varphi(x,y)$.
3.→4.: Is clear by compactness.
4.→5.: φ is symmetric, and by 4. it must be finite and reflexive, so φ is thick. Let $n<\omega$. We have to show that there is a sequence witnessing that φ is n-thick. By hypothesis, there is some $\psi\in nc(x,y)$ such that $\psi^k\vdash\varphi$, where $k=2^{n-1}$. Since $\psi^k\in nc^k(x,y)\subseteq nc^{k/2}(x,y)\subseteq nc(x,y)^{k/4}\subseteq\ldots\subseteq nc(x,y)$, there are $\psi_0,\ldots,\psi_{n-1}=\psi\in nc(x,y)$ such that $\psi^k=\psi_0=\psi_1{}^2=\psi_2{}^4=\ldots=\psi_{n-2}{}^{k/2}=\psi_{n-1}{}^k$. Hence, $\psi^k=\psi_i{}^l$, where $l=2^i$, for $i<n$. Then, inductively, we get $\psi_i=\psi^{k/l}$, with $l=2^i$. It follows that $\psi_{i+1}{}^2=\psi^{k/l}=\psi_i$, for all $i<n-1$. So trivially holds $\psi_{i+1}{}^2(x,y)\vdash\psi_i(x,y)$, for $i<n-1$. Furthermore, $\psi_0\vdash\varphi$. Hence, $\psi_1{}^2\vdash\varphi$. Then the sequence $(\varphi,\psi_1,\ldots,\psi_{n-1})$ witnesses that φ is n-thick.
5.→4. follows from Observation 2.8.25.5.
4.→1.: Suppose $\models\neg\varphi(a,b)$ for some pair a, b. By 4., $\neg nc^n(a,b)$, for all $n<\omega$. Then we have $\neg(\vee_n nc^n(a,b))$, whence $E_L(a,b)$ does not hold.
q.e.d.

Remark 2.8.31 : In the previous Proposition φ is required to be symmetric. But if $E_L(x,y)\vdash\varphi(x,y)$, with φ not necessariliy symmetric, then follows that $E_L(x,y)\vdash\varphi(x,y)\wedge\varphi(y,x)$ (since E_L is a symmetric relation) and this is clearly a symmetric formula which implies $\varphi(x,y)$.

Proposition 2.8.32 : $cl(E_L)$ is type-definable by the set of all formulas which are n-thick for all $n<\omega$.

Proof : By Proposition 2.8.30, we have for all symmetric formulas $\varphi(x,y)$: $E_L(x,y)\vdash\varphi(x,y)$ if and only if φ is n-thick for all $n<\omega$.
Now, the assertion follows from Observation 2.8.29. (Note that by Remark 2.8.31, it is sufficient to consider only symmetric formulas.)

Observation 2.8.33 : $E_L\subseteq cl(E_L)\subseteq E_{KP}$.

Proof : This follows immediately from the definitions.
q.e.d.

The following results (including Proposition 2.8.39) are due to the author of this thesis and was motivated by his studies supervised by Prof. E. Casanovas at the University of Barcelona.

Observation 2.8.34 :

1. E_L is type-definable if and only if $E_L=cl(E_L)=E_{KP}$.
2. If there exists some $m<\omega$ such that $\vee_n nc^n(x,y)\equiv nc^m(x,y)$, then $E_L=E_{KP}$.

Proof : 1. follows immediately from Propostition 2.8.10 and Observation 2.8.33.
Recall that $E_L=\vee_n nc^n(x,y)$. By the hypothesis of 2., E_L is type-definable by the set $\{\varphi^m : \varphi\in nc(x,y)\}$. So $E_L=E_{KP}$, by 1.
q.e.d.

Proposition 2.8.35 : The following conditions are equivalent:

1. $cl(E_L)=E_{KP}$.
2. For every formula $\varphi(x,y)$ holds:

   φ is n-thick for all $n<\omega$ if and only if φ is ω-thick.

Proof : Suppose p, q are partial types, defining $cl(E_L)$, E_{KP} respectively. By 2.8.27 and 2.8.32, we may assume that p is the set of all formulas which are n-thick for all $n<\omega$, and q is the set of all formulas which are ω-thick. If 1. holds, then p=q, and 2. follows. On the other hand, if 2. holds, then clearly p=q, and 1. folows.
q.e.d.

Proposition 2.8.36 : Let $0<m<\omega$. The following conditions are equivalent:

1. $\vee_n nc^n(x,y)\equiv nc^m(x,y)$.
2. $nc^m(x,y)$ contains only ω-thick formulas.

Proof : 1.→2.: First, we observe that if 1. holds, then $E_L(x,y)=\vee_n nc^n(x,y)$ is type-definable, namely by the set $\{\varphi^m : \varphi\in nc(x,y)\}$, hence $E_L=cl(E_L)=E_{KP}$, and by 2.8.25, every formula which is n-thick for all $n<\omega$ is ω-thick. Now suppose $\varphi\in nc^m(x,y)$, hence, by hypothesis, $E_L\vdash\varphi$. From Proposition 2.8.30 folllows that φ is ω-thick, so 2. holds.

2.→1.: We have to show that $\vee_n nc^n(a,b)$ implies $\models nc^m(a,b)$ (the other direction is obvious). So suppose there is some $n<\omega$ with $\models nc^n(a,b)$. If $n\leq m$, then the assertion follows from $nc^n(x,y)\supseteq nc^m(x,y)$. Now suppose $n>m$, and $\neg nc^m(a,b)$. Then $\models\neg\varphi(a,b)$, for some $\varphi\in nc^m(x,y)$. Since φ is ω-thick, for every $n<\omega$ there is a thick formula $\psi(x,y)$ with $\models\neg\psi^n(a,b)$. It follows that $\models\neg nc^n(a,b)$, for all $n<\omega$. Contradiction. Hence $\models nc^m(a,b)$.

q.e.d.

Remark : Note that for the implication 2.→1. it is sufficient that $nc^m(x,y)$ contains only formulas which are n-thick for all $n<\omega$.

Proposition 2.8.37 : Let $k<\omega$. The following is equivalent:

1. The set $A=\{n<\omega$: there is some formula φ such that φ is n-thick, but not n+1-thick$\}$ is finite and its maximal element is k.
2. If some formula is k+1-thick, then it is n-thick for all $n<\omega$.

Proof : 1.→2.: If φ is n-thick, for $n\geq k+1$, then it must be n+1-thick, since otherwise n would be an element of A, contradicting the choice of k. Now follows inductively that φ is n-thick for all $n<\omega$.

2.→1.: If A was infinite or its maximal element was >k, then there would exist an $n\geq k+1$, $n\in A$, and a formula φ such that φ is n-thick, but not n+1-thick. This contradicts 2.

q.e.d.

Propostion 2.8.38 : Let $0<m<\omega$ and let k be the smallest integer such that $2^k\geq m$.

1. Suppose that $nc^m(x,y)$ contains only formulas which are n-thick for all $n<\omega$. Then the set A in Proposition 2.8.37 is finite and for its maximal element l holds $l\leq k$.

2. Now suppose that the set A is finite and for its maximal element l holds l<k. Then $nc^m(x,y)$ contains only formulas which are n-thick for all $n<\omega$.

Proof : Suppose that the set A in Proposition 2.8.37 is infinite or its maximal element l is greater than k. Then $2^l>m$, and there is some formula φ such that φ is l-thick and not l+1-thick. It does not hold $E_L(x,y)\vdash\varphi(x,y)$ since otherwise φ would be n-thick for all $n<\omega$, by Proposition 2.8.30. So there are a, b such that $E_L(a,b)=\vee_n nc^n(a,b)$ and $\models\neg\varphi(a,b)$ hold. Since φ is l-thick, by 2.8.25.2, there is some thick formula ψ with $\models\neg\psi^j(a,b)$, where $j=2^{l-1}$. But $\psi^j\in nc^j(x,y)\subseteq nc^m(x,y)$, since $j=2^{l-1}\geq m$. So $\neg nc^m(a,b)$ holds.

Claim : For all $n<\omega$ holds $\neg nc^n(a,b)$.

Proof of the claim: It is clearly true for all $n\leq m$, since $nc^m(x,y)\subseteq nc^n(x,y)$. Now suppose n>m. Since $nc^m(x,y)$ contains only formulas which are n-thick for all $n<\omega$, by Observation 2.8.25.5, there is some thick formula ψ with $\models\neg\psi^n(a,b)$, whence $\neg nc^n(a,b)$. This proves the claim. q.e.d. Claim.

So we have $\neg\vee_n nc^n(a,b)$, hence $\neg E_L(a,b)$. This contradicts our hypothesis $E_L(a,b)$. Hence k is an upper bound of A.

q.e.d.1.

2.: Let $j=2^l$. We have j<m. Let φ be a thick formula with $\varphi^m\in nc^m(x,y)$. By Observation 2.8.25.6, φ^j is l+1-thick, whence n-thick for all $n<\omega$, by Proposition 2.8.37. Since j<m, we have $\varphi^j\vdash\varphi^m$, so φ^m is also n-thick for all $n<\omega$. This proves the assertion.

q.e.d.2.

Corollary 2.8.39 : Let $0<m<\omega$. The following conditions are equivalent:

1. $\vee_n nc^n(x,y)\equiv nc^m(x,y)$.
2. $nc^m(x,y)$ contains only ω-thick formulas.
3. The set $A=\{n<\omega$: there is some formula φ such that φ is n-thick, not n+1-thick$\}$ is finite.
4. There exists some $k<\omega$ with the property: If some formula is k-thick, then it is n-thick for all $n<\omega$.

Proof : Propositions 2.8.37, 2.8.38.
q.e.d.

The previous results are valid in any compete theory. In the following we shall study these subjects in the context of simple theories.

Proposition 2.8.40 : Let T be simple, and $a_0 \perp_A a_1$. Then the following are equivalent:

1. $Lstp(a_0/A)=Lstp(a_1/A)$.
2. There is a Morley sequence $(a_i : i<\omega)$ over A.
3. There is an A-indiscernible sequence $(a_i : i<\omega)$.

Proof : 2.→3. is trivial, and 3.→1. follows from Lemma 2.8.6. So assume $Lstp(a_0/A)=Lstp(a_1/A)$. By Theorem 2.6.15, there is a model **M** containing A such that $tp(a_0/\mathbf{M})=tp(a_1/\mathbf{M})$ and $a_0a_1 \perp_A \mathbf{M}$. We now construct an **M**-independent sequence $(a_i : i<\omega)$ such that $tp(a_ia_j/\mathbf{M})=tp(a_0a_1/\mathbf{M})$ for all $i<j<\omega$. We can clearly start with a_0a_1, since $a_0 \perp_M a_1$.
Suppose we have already found $(a_i : i<n)$ for some $n<\omega$, and let $p_i(x)$ be the conjugate of $tp(a_1/\mathbf{M}a_0)$ over $\mathbf{M}a_i$ for all $i<n$. By Corollary 2.6.10 there is an a_n realizing $\cup_{i<n}p_i$ with $a_n \perp_{\mathbf{M}} (a_i : i<n)$. Then $tp(a_ia_n/\mathbf{M})=tp(a_0a_1/\mathbf{M})$, and we are done.
$a_i \perp_A \mathbf{M}$, and $a_i \models tp(a_0/A)$, for all $i<\omega$. So **M**-independence is type-definable as "$D(a_i/\mathbf{M}a_j : j<i,\varphi,k) \geq D(a_0/A,\varphi,k)$ for all formulas φ and all $k<\omega$" (see Remark 2.4.3). Then any finite subset of the set of formulas expressing that there is an infinite M-indiscernible, M-independent sequence $(a_i' : i<\omega)$ in $tp(a_0/\mathbf{M})$, with $tp(a_0'a_1'/\mathbf{M})=tp(a_0a_1/\mathbf{M})$, is satisfiable by Ramsey's Theorem. By compactness, and conjugating $a_0'a_1'$ to a_0a_1, we are done.
q.e.d.

Proposition 2.8.41 : Let T be simple. Then for any sequence a and any sets $A \subseteq B$ there exists $b \perp_A B$ such that $Lstp(a/A)=Lstp(b/A)$.

Proof : Let $\mathbf{M} \supseteq A$ be a model such that $a \perp_A \mathbf{M}$ (see Proposition 2.4.19(ii) for the existence of such **M**). Choose any $b \models tp(a/\mathbf{M})$ with $b \perp_{\mathbf{M}} B$ (see 2.4.19(ii) again). Transitivity of non-forking implies $b \perp_A B$.

q.e.d.

Proposition 2.8.42 : Let T be simple.

1. If $a \perp_A b$, then $\models nc_A(a,b)$ if and only if Lstp(a/A)=Lstp(b/A).
2. The following conditions are equivalent:
   (i) Lstp(a/A)=Lstp(b/A).
   (ii) $nc_A(x,a) \cup nc_A(x,b)$ does not fork over A.
   (iii) $nc_A(x,a) \cup nc_A(x,b)$ is consistent.

Proof : 1.: Follows from Proposition 2.8.16 and Proposition 2.8.40.
2.(i)⇒(ii): By Proposition 2.8.41, we may choose a sequence c with Lstp(c/A)=Lstp(a/A) and $c \perp_A ab$. Now we use 1. to show that $\models nc_A(c,a) \cup nc_A(c,b)$. (ii)⇒(iii) is clear. (iii)⇒(i) by Lemma 2.8.6 and Proposition 2.8.26.
q.e.d.

By some of the previous results of the author we are now able to prove the following:

Corollary 2.8.43 : If T is simple, then equality of Lascar strong type is type-definable. More precisely, for every set A and tuples x, y of the same length, Lstp(x/A)=Lstp(y/A) is given by a partial type $r_A(x,y)$. Furthermore, $r_A = \cup\{r_a : a \in A, \text{ a is finite}\}$. So Lstp(x/A)=Lstp(y/A) if and only if Lstp(x/a)=Lstp(y/a) for all finite $a \in A$.

Proof : By Proposition 2.8.42, Lstp(a/A)=Lstp(b/A) if and only if $\models nc_A^2(a,b)$. So equality of Lascar strong type is type-definable by the set $\{\varphi^2(x,y) : \varphi \in nc_A(x,y)\}$. The rest of the assertion is clear.
q.e.d.

Corollary 2.8.44 : If T is simple, then the following hold:

1. $E_L = cl(E_L) = E_{KP}$.
2. $\vee_n nc^n(x,y) \equiv nc^2(x,y)$.
3. $nc^2(x,y)$ contains only ω-thick formulas.

4. Every formula which is n-thick for all $n<\omega$ is ω-thick.
5. If some formula is 2-thick, then it is ω-thick.

Proof : The assertions follow immediately from the previous results of this chapter.
q.e.d.

Definition 2.8.45 : Let a, b arbitrary sequences. We define $d_A(a,b)$ as the least $n<\omega$ with the property that there are sequences $a_0,\ldots,a_n$ and infinite sets $I_0,\ldots I_{n-1}$ such that $a_0=a$, $a_n=b$ and for every i, a_iI_i and $a_{i+1}I_i$ are A-indiscernibles. If there is no such n, we put $d_A(a,b)=\infty$.

Proposition 2.8.46 :

1. If $\mathbf{M}\supseteq A$ and $tp(a/\mathbf{M})=tp(b/\mathbf{M})$, then $d_A(a,b)=1$.
2. Lstp(a/A)=Lstp(b/A) if and only if $d_A(a,b)<\omega$.
3. If T is simple, then Lstp(a/A)=Lstp(b/A) if and only if $d_A(a,b)\leq 2$.

Proof : 1.: Let **N** be an $card(\mathbf{M})^+$-saturated elementary extension of **M**, and let $I\subseteq\mathbf{N}$ be an infinite coheir sequence of $tp(a/\mathbf{M})$ over $\mathbf{M}\cup\{a,b\}$. Then both aI and bI are coheir sequences over **M** and therefore **M**-indiscernible.
2.: The direction from left to right follows from 1. and the definition of Lascar strong type. For the other direction it is enough to show that $d_A(a,b)=1$ implies Lstp(a/A)=Lstp(b/A). This follows from Lemma 2.8.6 and the transitivity of equality of Lascar strong type.
3.: Let T be simple and Lstp(a/A)=Lstp(b/A). Choose c such that $c\perp_A ab$ and Lstp(c/A)=Lstp(a/A) (see 2.8.41). By Theorem 2.6.15, there is some model $\mathbf{M}\supseteq A$ with $tp(a/\mathbf{M})=tp(c/\mathbf{M})$. By 1., $d_A(a,c)=1$. At the same way, $d_A(c,b)=1$. Hence, $d_A(a,b)\leq 2$.
The other direction is clear, by 2.
q.e.d.

Remark 2.8.47 : The previous Proposition implies once more the type-definability of equality of Lascar strong types in simple theories, since $d_A(x,y)\leq 2$ can be expressed by a partial type over A.

2.9 Hyperimaginaries

Hyperimaginaries appear in simple theories as canonical bases (see [HKP]). The reason why we are interested in hyperimaginaries is essentially that they allow to reduce the concept of Lascar strong types to that one of strong types (types over algebraically closed sets), and moreover, that the elimination of hyperimaginaries implies the equivalence of Lascar strong types and strong types (see the discussion below). The elimination of hyperimaginaries (and the resulting equivalence of Lascar strong types with strong types) is considered as the main problem in simple theories.

We shall present only some definitions and results concerning hyperimaginaries, sufficient just for our objectives. Lemma 2.9.14 is due to the author and is a considerable improvement of a result of Casanovas given as Lemma 2.9.14'. As conclusions we get some of the main results, 2.9.16 and 2.9.17.

We would like to mention that there was developed an extensive theory of hyperimaginaries ([HKP],[LaP]), including a theory of forking of hyperimaginaries, showing that these objects in many aspects can be treated like elements of the model. However, the main objective of the theory of hyperimaginaries is to try to show that they can be eliminated. This could be proved for the subclass of supersimple theories [BPW], but remains open for simple theories in general.

In the following we shall treat equivalence relations E(x,y), that are type-definable by some type p(x,y). To simplify matters, we often identify E with the type which defines it.

Definition 2.9.1 : Let x, y be sequences of the same length α. A type-definable equivalence relation E(x,y) is *countable* if α and the partial type definaing α are countable; E is finitary if α is finite.

Lemma 2.9.2 : If E(x,y) is a type-definable equivalence relation, then for some index set I there are type-definable countable equivalence relations $\{E_i(x_i,y_i) : i \in I\}$, with $x_i \subseteq x$ and $y_i \subseteq y$ for all $i \in I$, such that $\models E(x,y)$ if and only if $\models E_i(x_i,y_i)$ for all $i \in I$.

Proof : Since E is a type-definable equivalence relation, and hence symmetric and transitive, compactness implies that for every formula $\varphi(x,y)\in E$ there is a formula $\varphi'(x,y)\in E$ such that $\varphi'(x,y)\wedge\varphi'(y,z)$ implies $\varphi(x,z)\wedge\varphi(y,z)$. For every formula $\varphi_i\in E$ let E_i be the closure of φ_i under the operation which assigns to φ the formula φ'. It is easy to see that E_i is reflexive, symmetric and transitive, and thus a type definable countable equivalence relation. Clearly $\wedge_{i\in I}E_i$ is equivalent to E.
q.e.d.

Definition 2.9.3 : Let E(x,y) be an equivalence relation, type-definable over $\emptyset$. If a is a tuple of the same length as x (and y), then we denote the class a modulo E by a/E and call such a quantity a *hyperimaginary* element of *type* E. An hyperimaginary is *countable* or *finitary* if E is countable or finitary, respectively.
If **M** is a model, then $\mathbf{M}^{heq}$ is **M** together with the collection of all countable hyperimaginaries modulo type-definable equivalence relations over $\emptyset$.

If e is an hyperimaginary, then by Aut(**C**/e) we mean the set $\{f\in Aut(\mathbf{C}) : f(e)=e\}$.

Observation 2.9.4 :

1. Every real or imaginary element is an hyperimaginary element. Every sequence of real or imaginary elements can be considered as an hyperimaginary. Every sequence of hyperimaginaries can be considered as an hyperimaginary.
2. Let A be any set (of hyperimaginaries). Then there is some hyperimaginary e such that Aut(**C**/A)=Aut(**C**/e).
3. Every automorphism of some model **M** extends uniquely to an automorphism of $\mathbf{M}^{heq}$.
4. By Lemma 2.9.2, we may restrict our attention to countable hyperimaginaries, since every hyperimaginary is equivalent to a set of countable hyperimaginaries (in the sense that any automorphism fixing the hyperimaginary also fix the set of countable hyperimaginaries).

Proof : The first observation of 1. is clear. If a=(a_i : i∈I), b=(b_i : i∈I) are sequences of imaginaries, then define E(a,b) if and only if a_i=b_i for all i∈I. Now, any automorphism which fixes a also fixes the hyperimaginary a/E.

If e=(e_i : i∈I) is a sequence of hyperimaginaries e_i=a_i/E_i, then e can be considerated as the hypermaginary (a_i : i∈I)/E, where E((b_i : i∈I),(c_i : i∈I)) if and only if E_i(b_i,c_i) for all i∈I.

2.: We may consider A as a sequence by enumerating its elements. Then the assertion follows from 1.

3.: Is clear, since for any automorphism f must hold f(a/E)=f(a)/E, for every hyperimaginary a/E.

q.e.d.

Definition 2.9.5 : Let A be a set (possibly containing hyperimaginaries).

1. The (hyperimaginary) *definable closure* of A, denoted dcl(A), is the set of all countable hyperimaginaries which are fixed under all A-automorphisms.
2. The (hyperimaginary) *algebraic closure* of A, denoted acl(A), is the set of all countable hyperimaginaries which have only finitely many images under A-automorphisms.
3. The (hyperimaginary) *bounded closure* of A, denoted bdd(A), is the set of all countable hyperimaginaries which have only boundedly many (i.e. less than κ) images under A-automorphisms.

We say that an hyperimaginary a/E is bounded, if it has bounded orbit in Aut(**C**), whence if a/E is in bdd(∅). This happens in particular if E is a bounded equivalence relation. For ex., all classes of E_{KP} are bounded hyperimaginaries. M. Ziegler [Z] recently has proved, using thick formulas, the other direction: If a/E is a bounded hyperimaginary, then E must be a bounded relation.

If we want to emphasize that we take the hyperimaginary definable or algebraic closure, we may indicate this by a superscript ^{heq}.

We shall call two sets A and B *interdefinable* (resp. *interalgebraic* or *interbounded*) if dcl(A)=dcl(B) (resp. acl(A)=acl(B) or bdd(A)=bdd(B)).

Remark 2.9.6 :

1. Every hyperimaginary is interdefinable with a sequence of countable hyperimaginaries.
2. If e is an imaginary element in bdd(A), then e is in acl(A).
3. If e is an uncountable hyperimaginary fixed under all A-automorphisms, we shall still say that $e \in dcl(A)$ (and similarly for the algebraic and bounded closures). In this sense, $A \subseteq dcl(A) \subseteq acl(A) \subseteq bdd(A)$.
4. If $a/E \in bdd(A)$ then for every formula $\varphi(x,y) \in E(x,y)$ there are some $\psi(x) \in tp(a/A)$ and some $n(\varphi) < \omega$ such that $\wedge_{0 \leq i < j < n(\varphi)}[\psi(x_i) \wedge \neg\varphi(x_i,x_j)]$ is inconsistent.

Proof : 1. is Observation 2.9.4.3 and follows from Lemma 2.9.2.

2.: If $e \notin acl(A)$, and e an imaginary element, then compactness implies that there are α many realizations of tp(e/A) for every ordinal α. Hence, $e \notin bdd(A)$.

4.: Suppose $\varphi \in E(x,y)$ and there is a sequence $(a_i : i<\omega)$ with $a_i \models tp(a/A)$ such that $\models \neg\varphi(a_i,a_j)$ for all $i<j<\omega$. Then compactness implies that there is such sequence of arbitrary cardinality λ. Hence, $\models \neg E(a_i,a_j)$ for all $i<j<\lambda$, and a/E can not be in the bounded closure of A.

5.: We may consider A as a sequence $(a_i : i<\lambda)$, enumerating its elements. Now, this sequence is interdefinable with some hyperimaginary $e=(a_i : i<\lambda)/E$, where

q.e.d.

We shall now define the notion of types of hyperimaginaries. This suggests to treat hyperimaginaries in the same way like imaginaries and it is possible in many occasions. However, hyperimaginaries are not elements of the model.

As we have seen in Observation 2.9.4.2, instead of sets it is sufficient to consider hyperimaginaries.

Definition 2.9.7 :

1. If p and q are two partial types with $p \vdash q$ and $q \vdash p$, we shall say that p and q are *equivalent*, and denote this by $p \equiv q$.

2. Let E be a type-definable equivalence relation. A *partial E-type* is a type p(x) which is invariant under E, i.e. whenever E(a,a') holds, then $\models$p(a) if and only if $\models$p(a'). A partial type over a hyperimaginary a/E is a partial type over some parameters A, such that for any a/E-automorphism f we have p(x)≡f(p(x)).

Lemma 2.9.8 : If p(x) is a partial type over a/E and a/E∈dcl(B), then there is a partial type p' over B equivalent to p.

Proof : Suppose p=p(x) is a partial type over a/E with parameters A. Define q(x) as

$$\exists X \models tp(A/B) \wedge p(x,X).$$

Clearly, p$\vdash$q, as A witnesses the existential quantifier. Conversely, if c$\models$q, there is A'$\models$tp(A/B) with $\models$p(c,A'). So there is a B-automorphism mapping A' to A; since a/E∈dcl(B), this is in fact an a/E-automorphism. Hence p(x,A')$\vdash$p(x,A)=p, and $\models$p(c).
q.e.d.

Definition 2.9.9 : Let e=a/E and d=b/F be two hyperimaginaries. For every formula φ(x,y) let $\varphi_{E,F}(x,y)$ be the partial type

$$\exists x'y'(E(x,x') \wedge F(y,y') \wedge \varphi(x',y')).$$

(That is, $\varphi_{E,F}(x,y)=\{\exists x'y'\psi_1(x,x') \wedge \psi_2(y,y') \wedge \varphi(x',y') : \psi_1 \in E, \psi_2 \in F\}$.)
Then we define *tp(e/d)* as the union of all partial types $\varphi_{E,F}(x,b)$ associated to all formulas φ(x,y) with the property $\models$φ(a',b') for some a', b' such that E(a,a') and F(b,b') hold.

Remark 2.9.10 : tp(e/d) in Definition 2.9.9 is a partial type over b. It is easy to see that the choice of another representative for d=b/F yields an equivalent type.
To say that an hyperimaginary a/E realizes a partial type p only makes sense if p is an E-type. Clearly, tp(e/d) is a partial E-type. The following Lemma shows that tp(e/d) is a partial E-type over the hyperimaginary d.

Lemma 2.9.11 : Let e, e', d be hyperimaginaries. Then tp(e/d)≡tp(e'/d) if and only if there is an automorphism which fix d (as an hyperimaginary, that is, setwise) and maps e to e'.

Proof : If tp(e/d)≡tp(e'/d), then e' realizes tp(e/d). As established in Remark 2.9.10, e and e' must be hyperimaginaries of the same type E. So we may assume that e=a/E, e'=a'/E, d=b/F.

Claim : The set $p(x,y)=E(x,a')\cup F(y,b)\cup[tp(a,b)=tp(x,y)]$ is consistent.

Proof : Trivially we have E(a,a) and F(b,b). Note that e' realizes tp(e/d), whence $\models\varphi_{E,F}(a',b)$ holds for every $\varphi\in tp(a,b)$. So for every $\varphi\in tp(a,b)$ there are u, v such that $E(u,a')\wedge F(v,b)\wedge\varphi(u,v)$ holds. Compactness then implies the consistency of p(x,y).

q.e.d. Claim.

If a_0, b_0 realize p(x,y), then $E(a_0,a')$ and $F(b_0,b)$ and there is an automorphism which maps a in a_0 and b in b_0. Clearly, this automorphism transforms e in e' and fixes d.

Conversely, suppose that there is such an automorphism f. If d=b/F, then tp(e/d) is a partial type over b. Hence, f(tp(e/d))=q is a partial type over f(b). q is equivalent to tp(e/d), since the definition of hyperimaginary types does not depend on the choice of the representatives. So it is sufficient to show that q is realized by e'=a'/E=f(e)=f(a/E)=f(a)/E. But this is clear, since the fact that $\varphi_{E,F}(x,b)$ is realized by a implies that $\varphi_{E,F}(x,f(b))$ is realized by f(a).

q.e.d.

Observation 2.9.12 :

Now, analogous to Lemma 2.2.10, it is easy to see that for hyperimaginaries e, d hold: $e\in dcl(d)$ if and only if the type tp(e/d) is realized only by e; and $e\in bdd(d)$ if and only if the type tp(e/d) has only a bounded number of hyperimaginary realizations.

From this observation it follows quickly that for any hyperimaginary e and any automorphism $f\in Aut(\mathbf{C})$ holds: f(dcl(e))=dcl(f(e)), and f(bdd(e))=bdd(f(e)).

Hyperiaginaries are classes of equivalence relations which are definable over the empty set. In order to prove a result which connects Lascar strong types to the concept of hyperimaginaries we need a slightly more general notion. The following definition and results was developed when the author of this thesis was studying under the orientation of Prof. Casanovas [Cas4] at the University of Barcelona. Lemma 2.9.14 is due to the author.

Definition 2.9.13 : An A-hyperimaginary is the equivalence class a/E of some sequence a and an equivalence relation E which is type-definable over the set A.

Every hyperimaginary is an A-hyperimaginary, but the converse is not true. However, we have been able to show that every A-hyperimaginary is equivalent to an hyperimaginary:

Lemma 2.9.14 : If e is an A-hyperimaginary, then there exists an hyperimaginary e' such that e and e' are interdefinable, that is, dcl(e)=dcl(e'). If e has bounded orbit under A-automorphisms, we may assume that e'$\in$bdd(A).

Proof : Let A be a set, a an enumeration of A, and let E_a=E(x,y;a) be an equivalence relation, type-definable over the set A. Let e=b/E_a be the class of all sequences which are E_a-equivalent to b. We define the relation E' by

$$E'(xz,yu) \text{ if and only if } (z \models tp(a) \wedge u \models tp(a) \wedge \forall v(E_z(x,v) \leftrightarrow E_u(y,v)) \vee xz=yu.$$

Claim: E' is an equivalence relation, type-definable over $\emptyset$.

Proof of the Claim: E' is clearly type-definable over $\emptyset$. It is easy to see that E' is reflexive and symmetric. We show symmetry. Suppose E'(cc',dd')$\wedge$E'(dd',ff'). The case cc'=dd' or dd'=ff' is easy to prove. So we may assume that c',d',f' $\models$tp(a), $\forall v(E_{c'}(c,v) \leftrightarrow E_{d'}(d,v))$ and $\forall v(E_{d'}(d,v) \leftrightarrow E_{f'}(f,v))$. Then follows $\forall v(E_{c'}(c,v) \leftrightarrow E_{f'}(f,v))$, whence, E'(cc',ff') holds, and E' is transitive. q.e.d. Claim.

Now we consider the hyperimaginary e'=ba/E'. We show that e' has the desired properties. Let f$\in$Aut(**C**) fix e and put b'a'=f(ba). Then holds E_a(b,b'). Since $E_{a'}$(b',b'), we obtain E'(ba,b'a'), and f fix e'. For the converse, suppose that f$\in$Aut(**C**) fix e', whence E'(ba,b'a'), where b'a'=f(ba). This means that b/E_a=b'/$E_{a'}$. Then f(e)=f(b/E_a)=b'/$E_{a'}$=b/E_a=e. So dcl(e)=dcl(e').

If e has a bounded number of images of A-automorphisms, then the same must hold for e'. In this case, if e' is a countable hyperimaginary, then e'$\in$bdd(A). If e' is not countable, then it is interdefinable with a sequence of countable hyperimaginaries (by Lemma 2.9.2), all elements of bdd(A).

q.e.d.

Remark : There is a weaker version of Lemma 2.9.14. We would like to quote it as Lemma 2.9.14'.

Lemma 2.9.14' : If e is an A-hyperimaginary, then there exists an hyperimaginary e' such that Aut(**C**/eA)=Aut(**C**/e'), that is, dcl(eA)=dcl(e'). If e has bounded orbit under A-automorphisms, we may assume that $e' \in bdd(A)$.

Proof : Let e=b/E, where E is an over A type-definable equivalence relation. Let a be a sequence enumerating A and put E=E(x,y;a). We define

$$E'(xz,yu) \text{ if and only if } (z=u \models tp(a) \wedge E(x,y;z)) \vee xz=yu.$$

This is an over $\emptyset$ type-definable equivalence relation. It is easy to see that for e'=ba/E' holds Aut(**C**/eA)=Aut(**C**/e'). If e has a bounded number of images of A-automorphisms, then the same must hold for e'. In this case, if e' is a countable hyperimaginary, then $e' \in bdd(A)$, otherwise e' is interdefinable with a sequence of countable hyperimaginaries (by Lemma 2.9.2), all elements of bdd(A).

q.e.d.

Proposition 2.9.15 : For every hyperimaginary e, the relation

$$F(x,y) \text{ if and only if } tp(x/e)=tp(y/e)$$

is type-definable over any representative of e.

Proof : Let e=a/E. Consider the partial type over a defined by $\exists u(E(a,u) \wedge tp(xa)=tp(yu))$. This type clearly says that there is an automorphism fixing e and mapping x to y. So it defines F(x,y).

q.e.d.

Lemma 2.9.16 : $E_{KP}^{A}(a,b)$ if and only if tp(a/bdd(A))=tp(b/bdd(A)).

Proof : Enumerating bdd(A) as a sequence of (countable) hyperimaginaries it can be coded into an hyperimaginary e (Observation 2.9.4). The relation E(x,y) defined by E(a,b) if and only if tp(a/bdd(A))=tp(b/bdd(A)) is bounded, since there are at most $2^{T+card(bdd(A))+card(x)}$ many types tp(x/bdd(A)). It is type-definable over some representative of e, by Proposition 2.9.15. Since E(x,y) is A-invariant, it is type-definable over A (see the Fact following Definition 2.2.1). Hence, $E_{KP}{}^{A}\subseteq E$.

For the other direction let us suppose that E(a,b) holds. Consider $e=a/E_{KP}{}^{A}$. This is an A-hyperimaginary with a bounded orbit of A-automorphisms, since $E_{KP}{}^{A}$ is bounded. By Lemma 2.9.14, there is an hyperimaginary e' with dcl(e')=dcl(e), and we may assume that e'∈bdd(A). By hypothesis, there is an automorphism f fixing bdd(A) and mapping a to b. Hence f fix e' and therefore f fix e (e∈dcl(e')). This means that $e=a/E_{KP}{}^{A}=b/E_{KP}{}^{A}$, whence $E_{KP}{}^{A}(a,b)$. Whence $E\subseteq E_{KP}{}^{A}$.

q.e.d.

Corollary 2.9.17 : If T is a simple theory, then

Lstp(a/A)=Lstp(b/A) if and only if tp(a/bdd(A))=tp(b/bdd(A)).

Proof : In an arbitrary theory Lstp(a/A)=Lstp(b/A) holds if and only if $E_L{}^{A}(a,b)$, by 2.8.20. If the theory is simple, then, by Corollary 2.8.43, E_L is type-definable, that is $E_L=E_{KP}$. Now the assertion follows from the previous Lemma.

Q.e.d.

In the following we shall treat equivalence relations which are restricted to a complete type. If R is a relation between I-sequences, x is an I-sequence of variables and p(x) a type, then $R\lceil p$ will denote the restriction of R to p(**C**), that is, $R\lceil p=\{(a,b)\in R : a\models p \text{ and } b\models p\}$.

Definition 2.9.18 : We say that T eliminates hyperimaginaries (or T has elimination of hyperimaginaries) if every hyperimaginary e is equivalent to a sequence of imaginaries, that is, there exists a sequence of imaginaries $(e_i : i\in I)$ such that $dcl(e)=dcl(e_i : i\in I)$.

Proposition 2.9.19 : T eliminates hyperimaginaries if and only if for every type $p(x)\in S(\emptyset)$ and every type-definable (over ∅) equivalence relation E between realizations of p there is a family $(E_i : i\in I)$ of definable equivalence relations such that $E=(\bigcap_{i\in I}E_i)\lceil p$. In fact, it is sufficient to consider definable relations E_i such that their restrictions $E_i \lceil p$ are equivalence relations.

Proof : First, we give the reason why it is sufficient to consider definable relations E_i whose restriction on p are equivalence relations: In this case we have $p(x) \vdash E_i(x,x)$, $p(x) \cup p(y) \vdash E_i(x,y) \rightarrow E_i(y,x)$, and $p(x) \cup p(y) \cup p(z) \vdash E_i(x,y) \wedge E_i(y,z) \rightarrow E_i(x,z)$. So, by compactness, there is some formula $\varphi_i \in p(x)$ with these properties. If we define: $F_i(x,y)$ if and only if $\varphi_i(x) \wedge \varphi_i(y) \wedge E_i(x,y)$, then $F_i(x,y)$ is a definable equivalence relation, and $(\cap_{i \in I} E_i) \lceil p = (\cap_{i \in I} F_i) \lceil p$.

The direction from right to left is clear: Suppose that $e=a/E$ is an hyperimaginary, and in $p(x)=tp(a)$ the equivalence relation E coincides with the intersection $\cap_{i \in I} E_i$ of definable equivalence relations E_i. Then $dcl(e)=dcl(e_i : i \in I)$, where $e_i=a_i/E_i$, and a_i is the corresponding finite subsequence of a.

For the other direction suppose that $E(x,y)$ is an equivalence relation in $p(x) \in S(\varnothing)$, type-defined over $\varnothing$. Let $a \models p$. By hypothesis, the hyperimaginary $e=a/E$ is interdefinable with some sequence $(e_i : i \in I)$ of imaginaries $e_i=a_i/E_i$.

Claim : Let $q(x,(y_i : i \in I))=tp(a,(a_i : i \in I))$. Then the following hold:

1. $E'(x,x') \cup q(x,(y_i : i \in I)) \cup q(x',(z_i : i \in I)) \vdash E_j(y_j,z_j)$, for all $j \in I$.
2. There is some $j \in I$ such that: $\neg E(x,x') \cup q(x,(y_i : i \in I)) \cup q(x',(z_i : i \in I)) \vdash \neg E_j(y_j,z_j)$
3. $\models q(a,(a_i : i \in I))$
4. $p(x) \vdash \exists y(q(x,y))$

Proof of the Claim: First, we note that if $e=a/E$ is interdefinable with a sequence $(e_i : i \in I)$ of imaginaries $e_i=a_i/E_i$, then any hyperimaginary b/E, with $tp(b)=tp(a)=p(x)$, is interdefinable with some sequence $(d_i : i \in I)$ of imaginaries $d_i=b_i/E_i$, with $b_i \models tp(a_i)$. This follows immediately from Observation 2.9.12.

1.: $q(b,(b_i : i \in I)) \cup q(c,(c_i : i \in I))$ says that there is an automorphism f mapping b to c and b_i to c_i for all $i \in I$. So f maps b/E to c/E. If additional $E(b,c)$ holds, then $b/E=c/E$, that is, f fix b/E. Then f must fix all b_i/E_i, that is, $b_i/E_i=c_i/E_i$ must hold for all $i \in I$, since $dcl(b/E)=dcl((b_i/E_i : i \in I))$ and $dcl(c/E)=dcl((c_i/E_i : i \in I))$.

2.: If $\models E_i(y_i,z_i)$ for all $i\in I$, and Suppose there is an automorphism f mapping x to x' and $(y_i : i\in I)$ to $(z_i : i\in I)$, which is expressed by the type q, and additional $\models\neg E(x,x')$. Then, by interdefinabilty of x/E and $(y_i/E_i : i\in I)$, x'/E and $(z_i/E_i : i\in I)$, respectively, there must be at least one $i\in I$ such that $\models\neg E_i(y_i,z_i)$.

3.: is obvious.

4.: Let $b\models p(x)$. Then there is an automorphism mapping a to b. This automorphism maps a_i to some b_i, since $q(x,(y_i : i\in I))=tp(a,(a_i : i\in I))$. Then $q(b,(b_i : i\in I))=tp(a,(a_i : i\in I))$.

Q.e.d. Claim.

By compactness, there is some formula $\psi(x,y)\in q$ such that ψ satisfies 2. of the Claim, and for every $j\in I$ there is $\psi_j(x,y)\in q$ which satisfies 1. of the Claim. Put $\varphi_i(x,y)=\psi_i(x,y)\wedge\psi(x,y)$ for all $i\in I$. Then φ_i has the same properties 1.-4. as the type q. Now we define

$$F_i(y,z) \text{ if and only if } \exists uv(E_i(u,v)\wedge\varphi_i(y,u)\wedge\varphi_i(z,v)).$$

Clearly, $F_i(y,z)$ is a definable relation. It is easy to check that F_i is reflexive and symmetric. For transitivity suppose that $\models F_i(a,b)\wedge F_i(b,c)$. That means that there are u, v and u', v' such that $\models E_i(u,v)\wedge\varphi_i(a,u)\wedge\varphi_i(b,v)\wedge E_i(u',v')\wedge\varphi_i(b,u')\wedge\varphi_i(c,v')$. Since $\models E(b,b)\wedge\varphi_i(b,v)\wedge\varphi_i(b,u')$ we get $\models E_i(v,u')$, by 1. Then $\models E_i(u,v)\wedge E_i(v,u')\wedge E_i(u',v')$ yields $\models E_i(u,v')$. Since $\models\varphi_i(a,u)\wedge\varphi_i(c,v')$, we get $\models F_i(a,c)$, and F_i is transitive. So F_i is an equivalence relation.

Now suppose $\models E(b,c)$. By 1. and 4. follows that there are b_i, c_i such that $\models E_i(b_i,c_i)$ for all $i\in I$. Hence, $F_i(b,c)$ for all $i\in I$.

Now suppose $\models F_i(b,c)$ for $b, c\models p(x)$ and all $i\in I$. Then by 2. follows that $\models E(b,c)$.

Hence, $E=(\bigcap_{i\in I}F_i)\upharpoonright p$.

q.e.d.

Proposition 2.9.20 : Let A be a set. If T eliminates hyperimaginaries, then T(A) also eliminates hyperimaginaries.

Proof : Let e=a/E be a hyperimaginary in T(A). E is type-definable over A and e is an A-hyperimaginary in T. By Lemma 2.9.14', there is an hyperimaginary e' in T, such that dcl(eA)=dcl(e'). Since T eliminates hyperimaginaries, e' is equivalent to a sequence of imaginaries. Then in e is equivalent to this sequence in T(A).

q.e.d.

Proposition 2.9.21 : Let A be a set. If T eliminates hyperimaginaries, then T eliminates A-hyperimaginaries. (That is, every A-hyperimaginary is interdefinable with a sequence of imaginaries).

Proof : Let e=a/E be an A-hyperimaginary, where E=E(x,y;A) is type-definable over A. By Lemma 2.9.14, there is a hyperimaginary e' interdefinable with e. By hypothesis, e' is interdefinable with a sequence of imaginaries. This proves the assertion.
q.e.d.

2.10 Strong types and the "Lstp=stp" problem

In the Independence Theorem over a model (Corollary 2.6.9) we have seen that in simple theories two non-forking extensions of a type over a model can be amalgamated, that is, they have a common non-forking extension. Pillay and Kim [Kim1] found a generalization of the Independence Theorem over a model, namely the Independence Theorem for Lascar strong types (Theorem 2.6.16). By Corollary 2.9.17, we know that in simple theories a Lascar strong type of a sequence c over some set A is the same as the type of c over bdd(A). So in Theorem 2.6.16, now we see that the two types tp(b/AB) and tp(c/AC) are in fact non forking-extensions of Lstp(b/A)=tp(b/bdd(A))=tp(c/bdd(A))=Lstp(c/A), since they do not fork over A. Moreover, $b\perp_{bdd(A)}B$ if and only if $b\perp_{A}B$, and $c\perp_{bdd(A)}C$ if and only if $c\perp_{A}C$. This follows from Transitivity of non-forking and the fact that $X\perp_{Y}bdd(Y)$ for all X, Y (by the remark of Proposition 2.2.5, an indiscernible sequence over Y is also indiscernible over bdd(Y)). Hence, non-forking over A is equivalent to non-forking over bdd(A). So the Independence Theorem for Lascar strong types says that non-forking extensions of types of the form tp(a/bdd(A))=Lstp(a/A) can be amalgamated.
However, $bdd^{heq}(A)$ is a much more complicated object than $acl^{eq}(A)$. Its elements are not elements of the monster model. For this reason it is desirable to reduce types over sets A=bdd(A) (i.e., Lascar strong types) to types over algebraically closed sets A=acl(A), in the

sense that holds tp(a/acl(A)) $\models$Lstp(a/A). In this case, strong types and Lascar strong types are equivalent. It is easy to see that if T eliminates hyperimaginaries, then strong types and Lascar strong types are equivalent. The question whether this equivalence holds we shall call the Lstp=stp problem. We say that Lstp=stp holds, if Lascar strong types are equivalent to strong types. As a consequence of Lstp=stp, the Independence Theorem holds also for strong types over arbitrary sets (not only for Lascar strong types or types over models).

There are subtheories of simple theories, such as the stable theories (see below) and the supersimple theories [BPW], which eliminate hyperimaginaries. However, the problem whether all simple theories eliminate hyperimaginaries remains open. The weaker Lstp=stp problem is solved for the class of low simple theories [Bue2].

Proposition 2.10.1 : The following conditions are equivalent for a definable relation $R \subseteq \mathbf{C}^n$.

1. R has finite orbit in Aut(**C**/A).
2. R is a union of classes of an A-definable finite equivalence relation.
3. R is definable over $acl^{eq}(A)$.
4. R is definable over any model $\mathbf{M} \supseteq A$.

Proof : 1.→2.: Let R_1, … ,R_2 be the distinct A-conjugates of R. Consider the equivalence relation E defined by E(a,b) if and only if $\wedge_{1 \le i \le n}(a \in R_i \leftrightarrow b \in R_i)$. E is definable and A-invariant, whence it is definable over A (by the fact after Definition 2.2.1). It is clear that every R_i is an union of classes of E.

2.→3.: Let E be a finite equivalence relation and A-definable and suppose that R is the union of the classes a_1/E, … a_n/E. Assume that E is definable by φ(b,x,y) with $b \in A$. Consider the relation F(ux,wy) defined by F(ux,wy) if and only if [φ(u,x,y) is an equivalence relation in x, y, and u=w, and $\models$φ(u,x,y)] or [φ(u,x,y) is not an equivalence relation in x, y, and x, y, w are arbitrary]. It is a ∅-definable equivalence relation, so ba_i/F is an element of $\mathbf{C}^{eq}$. Furthermore it is algebraic over A since the formula $\exists x f_F(b,x)=z$ has only a finite number of realizations, (where f_F is the function of $\mathbf{C}^{eq}$ mapping from sort $i_=$ to sort i_F and satisfying $\models \forall xy(f_F(x)=f_F(y) \leftrightarrow F(x,y))$). Hence, $ba_i/F \in acl^{eq}(A)$ for i=1,…,n. Now we can define R by the formula $f_F(b,x)=ba_1/F \vee \ldots \vee f_F(b,x)=ba_n/F$.

3.→4.: It is clear that R is definable over every $\mathbf{M}^{eq}\supseteq A$, since $acl^{eq}(A)\subseteq\mathbf{M}^{eq}$. By Lemma 2.2.6, R is also definable in **C** over every $\mathbf{M}\supseteq A$.

4.:→1.: Let **M** be a model of cardinality κ=card(A)+card(T). If R has infinitely many A-conjugates, then it must have at least κ^+ many, by compactness. Let $(R_i : i<\kappa^+)$ these A-conjugates, pairwise distinct. The hypothesis implies that every R_i is also definable over all models containing A, in particular over **M**. But there are at most κ-many possible definitions with parameters en **M**. Hence, two of the conjugates must have the same definition, whence are equal. This contradicts our assumption about κ^+ many pairwise distinct A-conjugates, and R has only finitely many A-conjugates.

q.e.d.

Definition 2.10.2 : The *strong type* of a sequence a over a set A is defined by $stp(a/A)=tp(acl^{eq}(A))$.

Remark : A stationary type extends to a unique strong type, since any extension of a type over some set A to $acl^{eq}(A)$ must be a non-forking extension.

Proposition 2.10.3 : stp(a/A)=stp(b/A) if and only if for every A-definable finite equivalence relation E, $\models E(a,b)$.

Proof : If E(x,y) is an A-definable finite equivalence relation, then there are $b\in A$ and a $\emptyset$-definable relation F(ux,wy) such that E(a,a') if and only if F(ba,ba') (see the proof of Proposition 2.10.1). But then $[ba]_F\in acl^{eq}(A)$ and $stp(a/A)\vdash f_F(b,x)=[ba]_F$, whence $\models E(a,a')$ if stp(a/A)=stp(a'/A).

For the other direction, we have to show that $\models\varphi(a)\leftrightarrow\varphi(b)$ holds for all formulas over $acl^{eq}(A)$. By Proposition 2.10.1, every formula $\varphi(x)$ over $acl^{eq}(A)$ defines an union of classes of an A-definable finite equivalence relation E_φ. Now suppose $\models\varphi(a)$ holds. By hypothesis, $\models E_\varphi(a,b)$, whence $\models\varphi(b)$ holds. Similary, if $\models\varphi(b)$ holds, then $\models E_\varphi(b,a)$ implies $\models\varphi(a)$.

q.e.d.

Observation 2.10.4 : $E_{Sh}^A(a,b)$ if and only if stp(a/A)=stp(b/A).
This follows from the definition of E_{Sh} and Proposition 2.10.3.

Definition 2.10.5 : By the "Lstp=stp Problem" we mean the question whether for all sequences a, b of the same length and all sets A holds

Lstp(a/A)=Lstp(b/A) if and only if stp(a/A)=stp(b/A).

If this equivalence is true in T, then we write "Lstp=stp".

Proposition 2.10.6 : $E_{Sh}=E_{KP}$ if and only if for any $p(x)\in S(\emptyset)$ and any bounded type-definable equivalence relation E between realizations of p there is a family of definable equivalence relations (E_i : $i\in I$) such that $E=(\bigcap_{i\in I}E_i)\upharpoonright p$. It is sufficient to consider definable relations E_i whose restrictions $E_i\upharpoonright p$ are equivalence relations.

Proof : Suppose $\models E_{Sh}(a,b)$. By Observation 2.10.4, stp(a)=stp(b), in particular tp(a)=tp(b). Let p(x)=tp(a), and consider $E_{KP}\upharpoonright p$. By hypothesis, $E_{KP}\upharpoonright p$ is the intersection of a set of definable equivalence relations E_i, $i\in I$, between realizations of p. Since $E_{KP}\upharpoonright p$ is bounded, all E_i must be finite, by compactness. By Proposition 2.10.3 (or even, by definition of E_{Sh}) it follows that $\models E_i(a,b)$ for all $i\in I$, whence $\models E_{KP}(a,b)$.
For the other direction it is sufficient to show that $E_{Sh}(x,y)$ implies E(x,y). Then E must be equivalent to an intersection of definable equivalence relations on tp(x). But this is obvious, since $E_{KP}(x,y)$ implies E(x,y), and $E_{KP}=E_{Sh}$, by hypothesis.
Q.e.d.

Corollary 2.10.7 : If T eliminates hyperimaginaries, then $E_{Sh}=E_{KP}$ in T(A), for all sets A.

Proof : This follows immediately from Proposition 2.9.19 and Proposition 2.10.6.
q.e.d.

Corollary 2.10.8 : Suppose that $E_L=E_{KP}$ holds in T(A), for all sets A. If T eliminates hyperimaginaries, then Lstp=stp. In particular, if T is simple and eliminates hyperimaginaries, then Lstp=stp.

Proof : If the hypothesis holds, then $E_L=E_{KP}=E_{Sh}$ in T(A), for every A, by Corollary 2.10.7. Then from 2.10.3 and 2.10.4 follows the assertion. If T is simple, then $E_L^A=E_{KP}^A$ for all A, by 2.8.43, and the assertion follows by the first statement.

One can show the Corollary also comparing $acl^{eq}(A)$ and $bdd^{heq}(A)$:

If $E_L=E_{KP}$ in T(A) for every A, then Lstp(a/A)=Lstp(b/A) if and only if tp(a/bdd(A))=tp(b/bdd(A)), by 2.8.20 and 2.9.16. It is clear that Lstp(a/A)=Lstp(b/A) implies stp(a/A)=stp(b/A) (Recall the definition of Lascar strong type and note that for any model **M**, acl(A)⊆**M**, if A⊆**M**.) Now suppose that stp(a/A)=stp(b/A), then there is an automorphism f fixing $acl^{eq}(A)$ and mapping a to b. Let e∈bdd(A) be an hyperimaginary. By hypothesis, e is equivalent to a sequence of imaginaries, all of them elements of bdd(A). By compactness, these imaginaries are elements of $acl^{eq}(A)$. But f fix $acl^{eq}(A)$, whence it fixes e. Since the choice of e∈bdd(A) was arbitrary, f fixes bdd(A) and tp(a/bdd(A))=tp(b/bdd(A)).

q.e.d.

Corollary 2.10.9 : Suppose that $E_L=E_{KP}$ in T(A), for every set A. Then Lstp=stp if and only if for every type p(x)∈S(A) and every over A type-definable and A-bounded equivalence relation E between realizations of p there is a family (E_i : i∈I) of equivalence relations, definable over A, such that $E=(\bigcap_{i\in I}E_i)\restriction p$.

Proof : If $E_L=E_{KP}$, then equality of Lascar strong type is defined by E_{KP}, by Proposition 2.8.20. Hence, Lstp=stp if and only if $E_{KP}=E_{Sh}$ in T(A) for all sets A, by Observation 2.10.4. Now the Corollary follows from Proposition 2.10.6.

q.e.d.

2.11 Some subclasses of simple theories

In Definition 2.5.12 we have defined the supersimple theories. In this chapter we shall define further subclasses of simple theories and study some of their properties. We are interested mainly in the Lstp=stp problem, which is solved for some subclasses but not for simple theories in general. In particular, supersimple theories eliminate hyperimaginaries [BPW], whence Lstp=stp.

Stable theories, which were developed by Shelah in order to classify all models of a complete first-order theory, are the best investigated simple theories. In stable theories a type over a model or an algebraic closed set has a unique non-forking extension to any superset. Hence, the Independence Theorem (over a model or an algebraic closed set) holds trivially in stable theories. Furthermore, stable theories eliminate hyperimaginaries (see [Wag]) and are simple. Hence, from the previous chapter follows that Lstp=stp in stable theories.

There is a rather natural way to define further subclasses of simple theories, using the notion of a dividing chain (Definition 2.4.6). Buechler [Bue2] proved that some of these theories, the so-called low theories, satisfy Lstp=stp. For years there have been efforts by different researchers (Casanovas, Shami, Wagner, Buechler and others) to answer the question whether these and other simple theories eliminate hyperimaginaries or whether they satisfy Lstp=stp. It seems that these problems are rather complicated and so far remain open. However, the author of this thesis found a new approach to treat the Lstp=stp problem in simple theories which shall present in this chapter. Furthermore, the author was able to give new characterizations of lowness. Separating a property that we define as the independent dividing chain property, this leads to one of our main results: Every ω-categorical simple theory with the independent dividing chain property is low. (Casanovas posed the general problem whether ω-categorical simple theories are low in [CasWag] where he proved that ω-categorical short theories are low.) The foundation of these results is Theorem 2.11.30, an improvement of an early result due to Kim [Kim1].

Finally, we define a new rank that allows characterizing low and short theories and has nice properties with respect to Morley sequences in simple theories. This rank and the new characterizations of low theories are promising tools for future investigations regarding the Lstp=stp problem and the relationships between some subclasses of simple theories.

We start by treating stable theories. These are historically the most important (Shelah's Classification Theory) and best investigated class of simple theories. Our interest in stable theories is motivated by the fact that they have some nice properties with respect to type amalgamation. Non-forking extensions of types over algebraically closed sets and over models can be amalgamated without the need of independence of their domains. So the Independence Theorem in these cases holds in a very strong form.

Definition 2.11.1 : A theory is *λ-stable*, or *stable in λ*, if card(S(A))$\leq\lambda$ for all A of size λ. A theory is *stable* if it is stable in some infinite λ.

Lemma 2.11.2 : A stable theory is simple.

Proof : Suppose T is not simple. Fix a cardinal λ and let λ' be minimal with $\lambda^{\lambda'}>\lambda$. By Proposition 2.5.11, there is a formula φ which has the tree-property, and by Remark 2.4.5, there is a (φ,k)-tree A of height λ', such that every node has λ successors. We may assume that k=2. But that means that over A there are $\lambda^{\lambda'}$ many types (consisting of the formula φ in λ' different parameters) such that any set of k such types is inconsistent. So we may assume that k=2 to see that there are $\lambda^{\lambda'}$ many pairwise inconsistent types over A. But card(A)=$\lambda^{<\lambda'}=\lambda$. So T is not stable in λ.
q.e.d.

Theorem 2.11.3 : In a stable theory, a type over a model has a unique non-forking extension to any superset.

Proof : Suppose not, and let **M** be a model, $p\in S(\mathbf{M})$, and $p_1(x,B)$ and $p_2(x,B)$ be two non-forking extensions of p to a set $B\supseteq\mathbf{M}$. We may assume that T is countable, whence there is a countable elementary submodel and two non-forking extensions to a countable superset. So we may assume that **M**, B are countable. Let $(B_i : i<\lambda)$ be a Morley sequence in tp(B/**M**). By Corollary 2.6.10, for every $I\subseteq\lambda$ the partial type

$$p_I=(\cup_{i\in I}p_1(x,B_i))\cup(\cup_{i\notin I}p_2(x,B_i))$$

is consistent. But $p_1(x,B_i)$ and $p_2(x,B_i)$ are two different types for all $i\in I$, so they can not have the same realization. It follows that the family $\{p_I : I<2^\lambda\}$ is pairwise incompatible. Hence, there are $2^\lambda>\lambda$ types over the λ parameters $\mathbf{M}\cup(\cup_{i<\lambda}B_i)$. So T is not λ-stable.
q.e.d.

Corollary 2.11.4 : A stable theory is λ-stable for all λ with $\lambda^{card(T)}=\lambda$.

Proof : Let A be a set of size λ contained in a model **M** of size λ. Clearly it is sufficient to count types over **M**. For every $p\in S(\mathbf{M})$ there is an elementary submodel $\mathbf{M}_p$ of size at most card(T), such that p does not fork over $\mathbf{M}_p$ (by Local Character of non-forking). p is uniquely determined as the non-forking extension of $p\lceil\mathbf{M}_p$ (by the previous Theorem), and there are at most $2^{card(T)}$ types over $\mathbf{M}_p$. On the other side, there are at most $\lambda^{card(T)}$ many submodels of cardinality card(T). So we get $card(S(\mathbf{M}))\leq\lambda^{card(T)}+2^{card(T)}=\lambda$.
q.e.d.

Corollary 2.11.5 : A simple theory is stable if and only if every type over a model has a unique non-forking extension to any superset.

Proof : Left to right follows follows from Theorem 2.11.3. Suppose T is simple and every type over a model has a unique non-forking extension. Let A be a set of size λ. There is a model **M** of size λ containing A. The proof of the previous Corollary shows that $S(A)\leq S(\mathbf{M})\leq\lambda$. So T is λ-stable.
q.e.d.

Corollary 2.11.6 : A type over an arbitrary set in a stable theory has at most $2^{card(T)}$ non-forking extensions to any given superset.

Proof : Suppose $p\in S(A)$ and $A\subseteq B$. Let $A_0\subseteq A$ be such that p does not fork over A_0 and $card(A_0)\leq card(T)$ (Local Character of non-forking), and let $\mathbf{M}\supseteq A_0$ be a model of size card(T). Then any non-forking extension q of p over B can be extended non-forkingly to some $q'\in S(\mathbf{M}\cup B)$, and does not fork over **M**, by Transitivity of non-forking. So it is uniquely determined by $q'\lceil\mathbf{M}$. As there are only $2^{card(T)}$ types over **M**, the assertion follows.
q.e.d.

Corollary 2.11.7 : In a stable theory, every type over a model or over an algabraically closed set (in T^{eq}) is stationary, and hence definable.

Proof : The first part follows immediately from Propositions 2.6.11 and 2.6.12, together with Corollary 2.11.6. By Corollary 2.4.22, stationary types in simple theories are definable.
q.e.d.

Remark 2.11.8 : Corollary 2.11.7 (and Theorem 2..11.3) provides a very strong form of type amalgamation. It says that the Independence Theorem in stable theories holds over models and algebraically closed sets in a trivially form, independence of sets is not necessary: Let C be an algebraically closed set or a model in a stable theory. If A, B are supersets of C, and $p \in S(A)$ and $q \in S(B)$ are non-forking extensions with the same restriction to C, then $p \cup q$ is consistent and does not fork over C, as it is part of the (unique) non-forking extension of $p \restriction C$ to $A \cup B$.

Theorem 2.11.9 : Every type in a stable theory is definable.

Proof : Consider $p \in S(A)$, and let q be a non-forking extension of p to acl(A). Then q is definable over acl(A) (Corollary 2.4.22), since it is stationary. For any formula $\varphi(x,y)$ consider the disjunction (the conjunction) of the finitely many A-conjugates of $d_q\varphi$. This is a formula invariant under A-automorphisms. By the Fact after Definition 2.2.1, it is definable over A. So the disjunction and the conjunction of the A-conjugates of $d_q\varphi$ both are φ-definitions for p over A.
q.e.d.

Remark 2.11.10 : Since types over algebraically closed sets are stationary in stable theories, every non-forking extension of $p \in S(A)$ to $B \cup acl(A)$ is uniquely determined by its restriction to acl(A). Furthermore, every extension of p to acl(A) must be a non-forking extension, since no element of acl(A) can be in an infinite indiscernible sequence over A. If q, q' are two non-forking extensions of p to some superset $C \supseteq A$, then we may assume that C=acl(A). As q, q' have the same restriction p over A, there is an automorphism f fixing A and mapping q to q'.

(This automorphism permutes acl(A), and $\varphi(x,a)\in q$ if and only if $\models d_q\varphi(a)$ if and only if $f(\varphi(x,a))\in q'$ if and only if $\models f(d_q\varphi(a))$, for every formula $\varphi(x,y)$.) Hence, all (non-forking) extensions of p to acl(A) are A-conjugated each other. But the φ-definition $d_q\varphi$ is over acl(A) and has only finitely many A-conjugates. So if we take the disjunction in the last proof, then we get a formula $\psi(y)$ over A such that $\psi(a)$ holds for any a if and only if some non-forking extension of p to Aa contains $\varphi(x,a)$. Similarly, if we take the conjunction, we get a formula $\psi'(y)$ over A such that $\psi'(a)$ holds for any a if and only if all non-forking extensions of p to Aa contain $\varphi(x,a)$. This is a considerable improvement on Lemma 2.4.20, where we have only a partial type (and not a formula) with this property.

The proof of the following fact requires more theory on hyperimaginaries and the study of canonical bases in stable theories. One can find these matters in [Wag].

Fact 2.11.11 : A stable theory eliminates hyperimaginaries. In particular, Ltp=stp holds.

We now move on to define further subclasses of simple theories by the notion of dividing chain.

Definition 2.11.12 :

1. A theorie T is low if for every formula $\varphi(x,y)$ there is a natural number n_φ such that φ does not divide n_φ times.
2. T is superlow if for every formula $\varphi(x,y)$ there is $n_\varphi<\omega$ such that $\wedge_k\varphi(x,z)$ does not divide n_φ times for all $k<\omega$, where $z=y_0\ldots y_k$ and $\wedge_k\varphi(x,z)=\varphi(x,y_0)\wedge\ldots\wedge\varphi(x,y_k)$.
3. T is short, if no formula divides ω many times.
4. T is supershort, if for every formula $\varphi(x,y)$ there is no infinite dividing chain made of instances of conjunctions $\wedge_k\varphi(x,z)$ (for varying k) of φ.

If $\varphi(x,y)$ divides n times for every $n<\omega$, we say that φ divides arbitrarily often.

Observation 2.11.13 :

1. T is simple if and only if no formula divides ω_1 times, this follows from Proposition 2.4.10 and Lemma 2.5.10. So simplicity of a theorie is also definable in terms of a dividing chain.
2. If T is supersimple, then no formula divides ω times (see Remark 2.5.14). Hence, T is short. Moreover, T is also supershort: Suppose not, then there is a formula $\varphi(x,y)$ and a type $p=\{\varphi(x,b_i) : i<\omega\}$, and any finite subset $(b_i : i<n)$ is contained in a finite subset $(b_i : i<m)$, $n\leq m$, such that some conjunction $\varphi(x,b_m)\wedge\varphi(x,b_{m+1})\wedge\ldots\wedge\varphi(x,b_{m+k})$ divides over $(b_i : i<m)$. So this conjunction also divides over $(b_i : i<n)$. Hence, the type p divides over every finite subset of its domain, and T cannot be supersimple.
3. If T is superlow, then T is low and supershort. Both supershort and low theories are short. Short theories are simple. This follows immediately from the definitions.
4. There are examples of simple nonshort theories [Cas1], supersimple nonlow theories [CasKim], and low nonsupershort theories [CasWag].
5. Note that bounding every dividing chain by some $n_\varphi<\omega$ in low theories is strictly stronger than the absence of a dividing chain of length ω in short theories: If some $\varphi(x,y)$ divides arbitrary often, this does not imply that φ divides ω times, since the "dividing numbers" k may vary and we can not apply compactness to get an infinite dividing chain.

Remark 2.11.15 : Low theories contain all known natural examples including also the stable theories. (The proof of the last fact requires a deeper study of ranks in stable theories.[Bue1])

The following rank is considered in [Bue2] to define low theories.

Definition 2.11.14 : The rank $D(p,\varphi)$ is defined for a set of formulas $p=p(x)$ and formulas $\varphi=\varphi(x,y)$ by:

1. $D(p,\varphi)\geq 0$ if and only if p is consistent
2. $D(p,\varphi)\geq\alpha+1$ if and only if there is a tuple a such that $\varphi(x,a)$ divides over the domain of p and $D(p\cup\{\varphi(x,a)\},\varphi)\geq\alpha$
3. If λ is a limit ordinal, then $D(p,\varphi)\geq\lambda$ if and only if $D(p,\varphi)\geq\alpha$ for all $\alpha<\lambda$.

Observation 2.11.15 : The rank here defined differs from the D(p,φ,k)-rank in Definition 2.4.2 only by the fact that the "dividing number" k in D(p,φ) is not fixed and may vary. So clearly we have D(p,φ,k)≤D(p,φ) for all p, φ, k. Furthermore, it is also clear that D(p,φ)≤D(q,φ), if q⊆p. Hence, D(x=x,φ) is maximal among all D(.,φ).

Proposition 2.11.16 : Let p=p(x) be a type and φ=φ(x,y) a formula. The following is equivalent for all n<ω:

1. D(p,φ)≥n.
2. There is a dividing chain (a_i : i<n) in φ, consistent with p.

Proof : We show the assertion by induction on n. If n=0, then the assertion is clear.
So suppose it is true for some n<ω, and let D(p,φ)≥n+1. By definition, there is a tuple a such that φ(x,a) divides over dom(p), and D(p∧φ(x,a),φ)≥n. By induction hypothesis, there is a sequence (a_i : i<n) such that p∪{φ{x,a}∪{φ(x,a_i) : i<n} is consistent, and φ(x,a_i) divides over dom(p)∪{a}∪{a_j : j<i}, for all i<n. This yields a sequence of length n+1 with the desired properties.
Now suppose 2. for n+1. Then by induction hypothesis, D(p∧φ(x,a_0),φ)≥n (witnessed by the sequence (a_i : 0<i<n)). Furthermore, φ(x,a_0) divides over dom(p). Hence, D(p,φ)≥n+1.
q.e.d.

Observation 2.11.17 : Suppose that (a_i : i<ω) is a dividing chain in φ(x,y), consistent with p(x). Then D(p,φ)=∞.
(That is, D(p,φ) is not ordinal valued: D(p,φ)≥α for all ordinals α.)

Proof : D(p,φ)>D(p∪{φ(x,a_0)},φ)>D(p∪{φ(x,a_0),φ(x,a_1)},φ)>…>… is an infinite descending chain. If D(p,φ) were ordinal valued this would be a contradiction to the well ordering of ordinals.
q.e.d.

The following result is due to the author.

Proposition 2.11.18 : The following conditions are equivalent, for $p=p(x)$ a type and $\varphi=\varphi(x,y)$ a formula:

1. $D(p,\varphi)=\infty$.
2. There is some tuple b such that $\varphi(x,b)$ divides over dom(p), and $D(p\wedge\varphi(x,b),\varphi)=\infty$.
3. There exists a dividing chain $(b_i : i<\omega)$ in φ, consistent with p.

Proof : 1.→2.: Since $D(p,\varphi)\geq\alpha+1$ for all ordinals α, this is witnessed by tuples b_α, with the property

$$D(p\wedge\varphi(x,b_\alpha),\varphi)\geq\alpha, \text{ and } \varphi(x,b_\alpha) \text{ divides over dom(p)}.$$

We have that for every tuple c and every $f\in Aut(\mathbf{C}/dom(p))$, $D(p\wedge\varphi(x,c),\varphi)=D(p\wedge\varphi(x,f(c)),\varphi)$, since ranks are invariant under automorphisms. So if $D(p\wedge\varphi(x,b_\alpha),\varphi)\neq D(p\wedge\varphi(x,b_\beta),\varphi)$, then $tp(b_\alpha/dom(p))\neq tp(b_\beta/dom(p))$. Since the number of types over dom(p) is bounded, there are only boundedly many b_α. Hence, considering the map which assign every ordinal α some b_α with the above property, there must be some b_α with $D(p\wedge\varphi(x,b_\alpha),\varphi)\geq\beta$ for all ordinals β (and $\varphi(x,b_\alpha)$ divides over dom(p)).
2.→3.: Repeating the step 1.→2. ω times yields the disired sequence.
3.→1.: This is Observation 2.11.17.
q.e.d.

Proposition 2.11.19 : T is low if and only if $D(p,\varphi)<\omega$ for all types p and all formulas φ.

Proof : Suppose T is low and fix a formula $\varphi(x,y)$. Since there is some $n<\omega$ such that there exists no dividing chain of length n in φ, $D(x=x,\varphi)<n$, by Proposition 2.11.16. Then holds for all types p, $D(p,\varphi)\leq D(x=x,\varphi)<n$.
Now suppose that T is not low. There is a formula $\varphi(x,y)$ such that there are dividing chains of arbitrary finite length in φ. By Proposition 2.11.16, $D(x=x,\varphi)\geq n$ for all $n<\omega$, whence $D(x=x,\varphi)\geq\omega$.
q.e.d.

Proposition 2.11.20 : T is short if and only if $D(p,\varphi)<\infty$, for all p, φ.

Proof : Suppose $D(x=x,\varphi)=\infty$. Then by Proposition 2.11.18, there is an infinite dividing chain in φ, and T is not short. Now suppose T is not short. Then there is some formula φ and an infinite dividing chain in φ. By Observation 2.11.17, $D(x=x,\varphi)=\infty$.
q.e.d.

Casanovas has recently proved [CasWag], that $D(p,\varphi)$ has only countable ordinals values whenever it is ordinal valued:

Proposition 2.11.21 [CasWag] : If $D(p,\varphi)\geq\omega_1$, then $D(p,\varphi)=\infty$.

Proof : Let A=dom(p) and assume $D(p,\varphi)\geq\omega_1$. We construct inductively a descending chain $(I_n : n<\omega)$ of cofinal subsets I_n of ω_1 and a sequence $(m_n : n<\omega)$ of natural numbers in such a way that for each $n<\omega$ and each $i\in I_n$ we can find tuples $a_0^i,\ldots,a_n^i$ such that $D(p(x)\cup\{\varphi(x,a_k^i) : k\leq n\},\varphi)\geq i$, and $\varphi(x,a_k^i)$ m_k-divides over $A\cup\{a_j^i : j<k\}$ for all $k\leq n$. Once these sequence have been constructed, compactness implies the existence of an infinite sequence $(a_n : n<\omega)$ with the property that $p(x)\cup\{\varphi(x,a_n) : n<\omega\}$ is consistent and $\varphi(x,a_n)$ m_n-divides over $A\cup\{a_k : k<n\}$ for each $n<\omega$. Then we obtain $D(p,\varphi)=\infty$, see Observation 2.11.17.
Assume we have obtained I_k and m_k for all $k<n$. Let $i\in I_{n-1}$ and choose $i'\in I_{n-1}$ such that $i'\geq i+1$. By inductive hypothesis there are $(a_k^i : k<n)$ such that $D(p\cup\{\varphi(x,a_k^i) : k<n\},\varphi)\geq i'$ and $\varphi(x,a_k^i)$ m_k-divides over $A\cup\{a_j^i : j<k\}$. Put $p'=p\cup\{\varphi(x,a_k^i) : k<n\}$. Since $D(p',\varphi)\geq i+1$, there is a tuple a_n^i and a natural number m_n^i such that $D(p'\cup\{\varphi(x,a_n^i)\},\varphi)\geq i$ and $\varphi(x,a_n^i)$ m_n^i-divides over $A\cup\{a_k^i : k<n\}$. Now consider the map which assigns to every $i\in I_{n-1}$ and tuple a_n^i the natural number $m_n^i<\omega$ with the described properties (let m_n^i be minimal to make sure that this map is a function). As $card(I_{n-1})>\omega$, we can find a subset $I_n\subseteq I_{n-1}$ of cardinality ω_1 such that this map is constant in I_{n-1}, that is, for all $i, j\in I_n$, $m_n^i=m_n^j=:m_n$.
q.e.d.

Lemma 2.11.22 : Let T be simple. The following conditions are equivalent:

1. T is low.
2. For every formula $\varphi(x,y)$ there is some $k<\omega$ such that for all tuples a the following holds: If $\varphi(x,a)$ divides over $\emptyset$, witnessed by some indiscernible sequence $(a_i : i<\omega)$, then $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent. Moreover, we may choose $k=D(x=x,\varphi)+1$.
3. For every formula $\varphi(x,y)$ there is some $k<\omega$ such that for all sets A and all tuples a the following holds: If $\varphi(x,a)$ divides over A, witnessed by some A-indiscernible sequence $(a_i : i<\omega)$, then $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent.
4. For any formula $\varphi(x,y)$ there is some $k<\omega$ such that $D(p,\varphi,m)=D(p,\varphi,k)$ for all $m\geq k$, and all partial types $p=p(x)$.
5. For any formula $\varphi(x,y)$ there is some $k<\omega$ such that $D(x=x,\varphi,m)=D(x=x,\varphi,k)$ for all $m\geq k$.

Proof : 1.→2. Let $I=(a_i : i<\omega)$ and put $k=D(x=x,\varphi)+1$. Suppose $p(x)=\{\varphi(x,a_i) : i<k\}$ is consistent. But $\varphi(x,a_i)$ divides over $\{a_j : j<i\}$ for every $i<k$. This is witnessed by $I-\{a_j : j<i\}$. So p(x) is a dividing chain of length $k>D(x=x,\varphi)$. This contradicts Proposition 2.11.16.
2.→3.: If $I=(a_i : i<\omega)$ witnesses that $\varphi(x,a)$ divides over A, I also witnesses dividing over $\emptyset$.
3.→4. Suppose $p=p(x)$ is a type over some set A. In any case we have $D(p,\varphi,m)\geq D(p,\varphi,k)$, if $m\geq k$. We show inductively that $D(p,\varphi,m)\leq D(p,\varphi,k)$ holds for $m\geq k$. Clearly, $D(p,\varphi,m)\geq 0$, and $D(p,\varphi,k)\geq 0$. Now suppose $D(p,\varphi,m)\geq\alpha+1$. Then there is some tuple a such that $\varphi(x,a)$ m-divides over A, and $D(p\wedge\varphi(x,a),\varphi,m)\geq\alpha$. Then $\varphi(x,a)$ k-divides over A, since every m-contradictory A-indiscernible sequence is n-contradictory by hypothesis. By induction hypothesis, $D(p\wedge\varphi(x,a),\varphi,n)\geq\alpha$. Hence $D(p,\varphi,n)\geq\alpha+1$.
4.→5. is clear.
5.→1.: Suppose that for any formula φ there is such a number k as in 5., and assume T to be not low. Then by Proposition 2.11.19, there is some $\varphi(x,y)$ such that $D(x=x,\varphi)\geq n$ for all $n<\omega$. Hence, by Proposition 2.11.16, for every $n<\omega$ there is a dividing chain of length n in φ. Taking the maximal "dividing numbers" m_n of these chains we get $D(x=x,\varphi,m_n)\geq n$ for every $n<\omega$. So if we choose $n>D(x=x,\varphi,k)$ such that $m_n\geq k$, then $D(x=x,\varphi,m_n)>D(x=x,\varphi,k)$, a contradiction.

q.e.d.

Corollary 2.11.23 : Let T be low. For any formula φ(x,y) there is some k such that for all sets A and all tuples a the following holds:

φ(x,a) divides over A if and only if φ(x,a) k-divides over A.

Furthermore, we may choose k=D(x=x,φ)+1.

Proof : This follows immediately from Lemma 2.11.22.
q.e.d.

By our further results in this chapter, we will be able to prove that the condition in the preceding Corollary is not only a conclusion of lowness, but also implies lowness (Proposition 2.11.33). Hence, it is a convenient characterization of lowness.

Corollary 2.11.24 : If T is low, then for every formula φ(x,y) and any set A there is a partial type q(y) over A such that for all tuples a the following holds:

φ(x,a) divides over A if and only if ⊨q(a).

Proof : The partial type q(y) will express that there is an A-indiscernible sequence (y_i : i<ω) of tp(y/A) which is k-inconsistent in φ(x,y) (that is, {φ(x,y_i) : i<ω} is k-inconsistent), where k is the number in Corollary 2.11.23. (The proof of Lemma 2.11.22 shows that we can choose k=D(x=x,φ)+1.)
q.e.d.

Remark 2.11.25 : This result, due to Buechler [Bue2], says that dividing over any set A in a low theory is type-definable over A. It is the key for Buechler's proof (see 2.11.46) that Lstp=stp holds in low theories. We shall see later, that any theory in which dividing is type-definable satisfies Lstp=stp. This motivates the following definition:

Definition 2.11.26 : We say that dividing over A is type-definable over B in a theory T, if for all formulas $\varphi(x,y)$ there is a partial type $p(y)$ over B such that for all tuples a the following holds:

$$\varphi(x,a) \text{ divides over A if and only if } \models p(a).$$

If for all sets A, dividing over A is type-definable over A in T, we say that dividing is type-definable in T.

We now look at ω-categorical theories. Recall that a theory T is called ω-categorical, if all its models of cardinality ω are isomorph. ω-categoricity of a theorie T is equivalent to the fact that the number of n-types over any finite set is finite: $card(S_n(A))<\omega$ for all $n<\omega$ and all finite A. This is known as the Theorem of Ryll-Nardzewski and can be found in any textbook of model theory.

Proposition 2.11.27 [CasWag] : An ω-categorical short theory is low. More precisely, if T is ω-categorical and $\varphi(x,y)$ divides arbitrary often, then $\varphi(x,y)$ divides ω times.

Proof : Consider the tree where the nodes at level n are types $q(y_i : i<n)$, such that $(a_i : i<n) \models q$ implies $\varphi(x,a_i)$ divides over $(a_j : j<i)$ for all $i<n$, and $\{\varphi(x,a_i) : i<n\}$ is consistent. Let successor be extension of the type. Since $\varphi(x,y)$ divides arbitrarily often, the tree has arbitrarily long branches. By ω-categoricity it is finitely branching and hence has an infinite branch, by König's Lemma. But this means that $\varphi(x,y)$ divides ω times.
q.e.d.

Corollary 2.11.28 : A supersimple ω-categorical theory is low.

Proof : A supersimple theory is short. Now the assertion follows from the preceding Proposition.
q.e.d.

The following results are due to the author of the thesis.

Proposition 2.11.29 : If T is ω-categorical, then dividing over finite sets is type-definable in T.

Proof : Fix some formula $\varphi(x,y)$, and let A be some finite set. If T is ω-categorical, then there are only finitely many different types tp(a/A) of finite tuples a of the same length as y over A. Let $tp(a_0/A),\ldots,tp(a_{n-1}/A)$ be these types. For every $i<n$ we may choose some fixed k_i (for example the smallest one) with the property that if $\varphi(x,a_i)$ divides over A, then $\varphi(x,a_i)$ k_i-divides over A; and put $k_i=0$, if $\varphi(x,a_i)$ does not divide over A. Let b be any tuple of the same length as y. There is exactly one $i<n$ such that $b \models tp(a_i/A)$. $\varphi(x,b)$ divides over A if and only if $\varphi(x,a_i)$ k_i-divides over A, since dividing over A is invariant under A-automorphisms. Now let $k=\max(k_i : i<n)$ be the maximal element of the k_i. Hence, for any b it holds that if $\varphi(x,b)$ divides over A, then $\varphi(x,a_i)$ k_i-divides over A, so it k-divides over A. As $tp(b/A)=tp(a_i/A)$, this implies that $\varphi(x,b)$ k-divides over A.
Now let q(y) be a partial type over A, expressing that there is an A-indiscernible sequence $(y_i : i<\omega)$ of tp(y/A) such that $\{\varphi(x,y_i) : i<k\}$ is inconsistent. Then clearly, $\varphi(x,b)$ divides over A if and only if $\models q(b)$, for all tuples b.
q.e.d.

If T is simple, then $\varphi(x,a)$ divides over A if and only if for any (some) Morley sequence I in tp(a/A), $\{\varphi(x,a') : a'\in I\}$ is inconsistent. This well-known fact (see Lemma 2.5.10) was proved by Kim [Kim1] (and played an important role in his proof of equivalence of dividing and forking in simple theories). However, Kim's proof of this fact do not supply a number k such that the set $\{\varphi(x,a') : a'\in I\}$ is k-inconsistent.
The following Proposition, one of our main results, is a considerable improvement and stronger form of Kim's result. We shall use it later to prove our characterization of lowness of a theory.

Theorem 2.11.30 : Let T be simple. Let p=p(x) be a partial type over A, φ(x,y) a formula, and b some tuple. If p(x)∪{φ(x,b)} k-divides over A, then for any A-independent sequence I=(b_i : i<ω) of tp(b/A), φ(x,b_i) k-divides over A∪{b_j : j<i} for all i<ω.
(In particular, this holds if I is a Morley sequence of tp(b/A).)
Furthermore:

(i) q:={φ(x,b_i) : i<ω} is m-inconsistent, for m=D(x=x,φ,k)+1.

(ii) q':=p(x)∪{φ(x,b_i) : i<ω} is m'-inconsistent, for m'=D(p,φ,k)+1. (Here m'-inconsistency means that p(x)∧φ(x,b_{i0})∧…∧φ(x,$b_{im'-1}$) is inconsistent for any subset {b_{i0},…,$b_{im'-1}$}⊆I of size m'.)

(Note that m'≤m.)

Proof : If φ(x,b) k-divides over A, then this is witnessed by some A-indiscernible sequence J=(b^j : j<ω), and we may assume b^0=b. Let I=(b_i : i<ω) be an A-independent sequence of tp(b/A). For every i<ω consider an A-automorphism mapping b to b_i. This automorphism maps J to some A-indiscernible sequence I_i=(b_i^j : j<ω), with b_i^0=b_i. Since $b_i \perp_A$ {b_j : j<i}, there is a sequence I_i', an Ab_i-automorph image of I_i, such that I_i' is A{b_j : j<i}-indiscernible. (This follows immediately from Proposition 2.3.6.) Hence, φ(x,b_i) k-divides over A{b_j : j<i} witnessed by I_i', for every i<ω.
Clearly, q must be inconsistent: Otherwise this would be a k-dividing chain in φ, and compactness would yield a k-dividing chain of length ω_1, contradicting simplicity of the theory. To show m-inconsistency of q, we suppose that q is m-consistent and will obtain a contradiction:
Suppose there is a subset {b_{i0},…,b_{im-1}}⊆I of size m such that φ(x,b_{i0})∧…∧φ(x,b_{im-1}) is consistent. Then from the definition of D(.,φ,k)-rank follows that there is a chain D(x=x,φ,k)>D(φ(x,b_{i0}),φ,k)>…>D(φ(x,b_{i0})∧…∧φ(x,b_{im-1}),φ,k)≥0, witnessed by k-dividing of φ(x,b_{ij}) over A∪{b_{ik} : k<j} for every j<m. So D(x=x,φ,k)≥m=D(x=x,φ,k)+1, a contradiction.
In a similarly way, if q'(x)∧φ(x,b_{i0})∧…∧φ(x,$b_{im'-1}$) was consistent for some subset {b_{i0},…,$b_{im'-1}$}⊆I of size m', this would imply the existence of a chain D(p,φ,k)>D(p∧φ(x,b_{i0}),k)>…>D(p∧φ(x,b_{i0})∧…∧φ(x,$b_{im'-1}$),φ,k)≥0. This chain witnesses that D(p,φ,k)≥m'=D(p,φ,k)+1, a contradiction.

q.e.d.

Corollary 2.11.31 : Let T be simple. Let p=p(x,b) be a partial type over Ab, and suppose p divides over A. Then there exists an $m<\omega$ such that for every A-independent sequence $I=(b_i : i<\omega)$ of tp(b/A), the set $\cup_{i<\omega}p(x,b_i)$ is m-inconsistent (where m-inconsistency here means that $\cup_{j<m}p(x,b_{ij})$ is inconsistent for any subset $\{b_{i0},\ldots,b_{im-1}\}\subseteq I$ of size m).

Proof : If p divides over A, there is a formula φ(x,b) implied by p(x,b) such that φ k-divides over A for some $k<\omega$. Put $m=D(x=x,\varphi,k)+1$. By the "furthermore clause" of the preceding Theorem, $\{\varphi(x,b_i) : i<\omega\}$ is m-inconsistent. It follows that $\cup_{i<\omega}p(x,b_i)$ is m-inconsistent.
q.e.d.

Our next result characterizes lowness of a theory similarly to Lemma 2.11.22. However, in Lemma 2.11.22 only those indiscernible sequences that witness dividing of φ are considered. Here we use Morley sequences instead of indiscernible sequences, but in fact, we consider all Morley sequences (of the same type).

Proposition 2.11.32 : Let T be simple. The following conditions are equivalent:

1. T is low.
2. For every formula φ(x,y) there is some $k<\omega$ such that for all sets A and all tuples a the following holds: If φ(x,a) divides over A, then $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent for *all Morley sequences* $(a_i : i<\omega)$ of tp(a/A).
3. For every formula φ(x,y) there is some $k<\omega$ such that for all *finite* sets A and all tuples a the following holds: If φ(x,a) divides over A, then $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent for *all Morley sequences* $(a_i : i<\omega)$ of tp(a/A).

Proof : 1.→2. : Suppose T is low. Put $k=D(x=x,\varphi)+1$. If φ(x,a) divides over A, then $\{\varphi(x,a_i) : i<\omega\}$ is inconsistent for all Morley sequences $(a_i : i<\omega)$ of tp(a/A), by Corollary 2.11.31. Hence, every Morley sequence of tp(a/A) witnesses that φ(x,a) divides over A. By Lemma 2.11.22, every Morley sequence witnesses k-dividing of φ(x,a) over A.

2.→3. is clear.

3.→1. : Suppose that 3. holds and assume that T is not low. Then there is some formula $\varphi(x,y)$ which divides arbitrarily often, i.e. for every $n<\omega$ there is a dividing chain $(\varphi(x,a_i) : i<n)$. But if $\varphi(x,a_i)$ divides over $\{a_j : j<i\}$ for all $i<n$, then $\varphi(x,a_i)$ k-divides over $\{a_j : j<i\}$ for all $i<n$, witnessed by any Morley sequence J of $tp(a_i/\{a_j : j<i\})$. In other words, every dividing chain in φ of finite length is a k-dividing chain, for some fixed $k<\omega$. So there are k-dividing chains of arbitrary finite length in φ. Then compactness implies the existence of a dividing chain in $\varphi(x,y)$ of length ω_1. This contradicts simplicity of T. Hence, T is low.

q.e.d.

As a new result we get a further and good readable characterization of lowness. In Corollary 2.11.23 we have seen that lowness implies for every formula the existence of a number which witnesses dividing of φ over all sets and for all tuples. On the other hand, the existence of such a number implies lowness.

Proposition 2.11.33 : Let T be simple. The following conditions are equivalent:

1. T is low.
2. For every formula $\varphi(x,y)$ there exists a $k_\varphi<\omega$ such that for all sets A and all tuples a holds:

   $$\varphi(x,a) \text{ divides over } A \leftrightarrow \varphi(x,a)\ k_\varphi\text{-divides over } A.$$

3. For every formula $\varphi(x,y)$ there exists a $k_\varphi<\omega$ such that for all *finite* sets A and all tuples a holds:

   $$\varphi(x,a) \text{ divides over } A \leftrightarrow \varphi(x,a)\ k_\varphi\text{-divides over } A.$$

Proof : 1.→2. follows from Corollary 2.11.23.

2.→3.: is clear.

3.→1.: To show lowness of T, we prove that the condition 3 of Proposition 2.11.32 holds. So fix a formula $\varphi(x,y)$ and put $m=D(x=x,\varphi,k_\varphi)+1$. Suppose that $\varphi(x,a)$ divides over some finite set A. By hypothesis and by Theorem 2.11.30, for every Morley sequence I in $tp(a/A)$, the set

$\{\varphi(x,b) : b \in I\}$ is m-inconsistent. Since the tuple a and the set A was arbitrary, Proposition 2.11.32 says that T is low.
There is another way to show that T is low: If 3. holds, then by induction it is easy to see that $D(p,\varphi,k_\varphi) \geq D(p,\varphi)$ (whence, $D(p,\varphi,k_\varphi)=D(p,\varphi)$) for all formulas φ and all types p over *finite* sets A. In particular, $D(x=x,\varphi)<\omega$. Then T is low, by Proposition 2.11.19.
q.e.d.

It seems that in general, in Propositions 2.11.32 and 2.11.33, respectively, for the third condition it is necessary to consider *all* finite sets A: We were not able to derive lowness of the theory from this condition considering dividing only over the empty set $A=\emptyset$. To change this, we introduce the following property:

Definition 2.11.34 : We say that a formula φ has the *independent dividing chain property* (or *φ has idcp)* if for every ordinal α and every dividing chain of length α in φ there is an independent dividing chain $(a_i : i<\alpha)$ of length α in φ, i.e. $a_i \perp (a_j : j<i)$ for all $i<\alpha$. We say that a theory T has the *independent dividing chain property* (or *T has idcp*) if every formula has idcp.

Theorem 2.11.35 : Let T be simple and suppose that T has idcp. The following conditions are equivalent:

1. T is low.
2. For every formula $\varphi(x,y)$ there is some $k<\omega$ such that for all tuples a the following holds: If $\varphi(x,a)$ divides over $\emptyset$, then $\{\varphi(x,a_i) : i<\omega\}$ is k-inconsistent for *all Morley sequences* $(a_i : i<\omega)$ of tp(a).
3. For every formula $\varphi(x,y)$ there exists a $k_\varphi<\omega$ such that for all tuples a holds:

$$\varphi(x,a) \text{ divides over } \emptyset \leftrightarrow \varphi(x,a)\ k_\varphi\text{-divides over } \emptyset.$$

Proof : 1.→3.: follows from Lemma 2.11.33.

3.→2.: Put k=D(x=x,φ,k_φ)+1. By the hypothesis of 3., we have for all a, if φ(x,a) divides over $\emptyset$, then φ(x,a) k_φ-divides over $\emptyset$. From Theorem 2.11.30 follows that {$\varphi(x,a_i)$: i<ω} is k-inconsistent for any Morley sequence (a_i : i<ω) of tp(a).

2.→1.: Suppose that 2. holds. If T is not low, then there is some φ(x,y) that divides n times for all n<ω. If $\varphi(x,a_i)$ divides over {a_j : j<i) for i<n, and J is a Morley sequence of tp(a_i/(a_j : j<i)), then J is a Morley sequence of tp(a_i): Indiscernibility over $\emptyset$ is clear, and indepedence over $\emptyset$ follows from $a_i \perp (a_j : j<i)$ and transitivity of the independence relation. By 2., {φ(x,b) : b$\in$J} is k-inconsistent and J witnesses that $\varphi(x,a_i)$ k-divides over (a_j : j<i). Hence, every dividing chain in φ is a k-dividing chain. Then there are dividing k-chains of arbitrary finite length in φ, and compactness implies the existence of a dividing chain of length ω_1. This contradicts simplicity of T.

q.e.d.

Theorem 2.11.36 : Let T be simple and ω-categorical. If T has idcp, then T is low.

Proof : We show that T satisfies the third condition of Theorem 2.11.35 for any formula φ(x,y). So fix some formula φ(x,y), and let n be the length of the tuple y. Since T is ω-categorical, there are only finitely many n-types over $\emptyset$. Let tp(a_0), … ,tp(a_{m-1}) be these types. Let k_i be the smallest number such that $\varphi(x,a_i)$ k_i-divides over $\emptyset$, and put k_i=0, if $\varphi(x,a_i)$ does not divide. Let k=max{k_i : i<m}. Then for every tuple a (of length n) holds, a $\models$tp(a_i) for exactly one i<m, and therefore: If φ(x,a) divides over $\emptyset$, then φ(x,a) k-divides over $\emptyset$. From Theorem 2.11.35 follows that T is low.

q.e.d.

Now we define a new rank, which allows characterizing low and short theories.

Definition 2.11.37 : Let p=p(x) be a set of formulas in the variable x over A, φ=φ(x,y) a formula, I=(a_i : i<β) an infinite sequence. We define the rank D(p,φ,I) as follows:

1. D(p,φ,I)$\geq$0, if p(x) is consistent.

2. $D(p,\varphi,I)\geq\alpha+1$, if $D(p\cup\{\varphi(x,b)\},\varphi,I_1)\geq\alpha$ and $\varphi(x,b)$ divides over A, where b is the first element of I, and I_1 is the sequence I-{b}.
3. $D(p,\varphi,I)\geq\lambda$, for λ a limit ordinal, if $D(p,\varphi,I)\geq\beta$ for all $\beta<\lambda$.

Observation 2.11.38 :

1. $D(p,\varphi,I)\leq D(p,\varphi)$ for all p, φ, I.
2. For all $\alpha\leq\omega$: $D(p,\varphi,I)\geq\alpha$ if and only if the first α many elements of I form a dividing chain in φ, consistent with p.
3. If T is simple and $I=(a_i : i<\omega)$ is an A-independent sequences of $tp(a_0/A)$, then $D(p,\varphi,I)<\omega$ for all partial types p over A, and all formulas $\varphi(x,y)$. If $\varphi(x,a_0)$ k-divides over A, then $D(p,\varphi,I)\leq D(p,\varphi,k)$. In particular, this holds if I is a Morley sequence over A.
4. T is low if and only if for every formula $\varphi(x,y)$ there is some $n_\varphi<\omega$ such that $D(x=x,\varphi,I)\leq D(x=x,\varphi)=n_\varphi$ for all sequences I.
5. T is short if and only if for every formula $\varphi(x,y)$, $D(x=x,\varphi,I)<\omega$ for all sequences I.

Proof : 1. is clear.

2.: We show the assertion by induction. The assertion is trivial for $\alpha=0$. Suppose it holds for some α, with $0<\alpha<\omega$. Then: $D(p,\varphi,I)\geq\alpha+1 \leftrightarrow D(p\wedge\varphi(x,a),\varphi,I_1)\geq\alpha$ and $\varphi(x,a)$ divides over dom(p), where a is the first element of I, and $I_1=I-\{a\}$ $\leftrightarrow$ the first α elements of I_1 form a dividing chain in φ, consistent with p $\leftrightarrow$ the first $\alpha+1$ elements form a dividing chain in φ, consistent with p.

Now suppose the assertion holds for all $\beta<\omega$. Then: $D(p,\varphi,I)\geq\omega \leftrightarrow D(p,\varphi,I)\geq\alpha$ for all $\alpha<\omega$ $\leftrightarrow$ every finite starting subsequence of I forms a dividing chain in φ, consistent with p $\leftrightarrow$ $(a_i : i<\omega)$ is a dividing chain in φ, consistent with p, where $I=(a_i : i<\lambda)$ for some ordinal $\lambda\geq\omega$.

3.: This follows from Theorem 2.11.30.

4.: If T is low $\leftrightarrow$ every dividing chain in φ is bounded by some $n_\varphi=D(x=x,\varphi)<\omega$.

5.: T is short $\leftrightarrow$ no formula divides arbitrary often $\leftrightarrow$ for all formulas φ, every dividing chain in φ is finite.

q.e.d.

The general idea to prove Lstp=stp in a simple theory is clear (note that, trivially, Lascar strong type implies strong type): Equality of Lascar strong types is type definable in simple theories (see chapter 2.8). Since stp(a/A)=stp(b/A) implies $\models E(a,b)$ for all finite A-definable equivalence relations E, it is sufficient to show that equality of Lascar strong type, defined by some partial type p(x,y) over A, is equivalent to a conjunction of finite A-definable equivalence relations.

On the other hand, by the results of chapter 2.8, we know that in simple theories equality of Lascar strong types is type-definable by the set of all ω-thick formulas. So if we were able to show that every ω-thick formula is implied by some definable finite equivalence relation, then we would conclude that Lascar strong type is implied by strong type, whence Lstp=stp. This idea motivates the following definition and results, due to the author of this thesis.

Definition 2.11.39 : Let $\varphi(x,y)$ be a thick formula. We define $\psi_\varphi(x,y):=\forall z(\varphi(x,z)\leftrightarrow\varphi(y,z))$.

The following Observation is easy to prove:

Observation 2.11.40 : For every thick formula φ, ψ_φ defines an equivalence relation and implies φ.

Theorem 2.11.41 : Suppose that in T holds $E_L^A=E_{KP}^A$ for all sets A. If for every ω-thick formula φ, the formula ψ_φ is finite, then Lstp=stp in T.

Proof : Fix some set A and suppose stp(a/A)=stp(b/A). Then $\models E(a,b)$ for all A-definable finite equivalence relations E. In particular $\models\psi_\varphi(a,b)$ for all ω-thick φ. So we obtain $\models\varphi(a,b)$ for all ω-thick formulas φ over A. Since $E_L^A=E_{KP}^A$, equality of Lascar strong types over A is type-definable over A, whence it is type-definable by the set of all ω-thick formulas over A (Propositions 2.8.20 and 2.8.27). It follows that Lstp(a/A)=Lstp(b/A). Since A was arbitrary, Lstp=stp.

q.e.d.

The previous Theorem leads to the question of under which conditions the formulas ψ_φ define finite equivalence relations. The following Proposition give some answers.

Proposition 2.11.42 : Let $\varphi(x,y)$ be a thick formula with parameters in A. The following are equivalent:

1. ψ_φ defines a finite equivalence relation.
2. φ is invariant under strong types over A (that is, if $\models\varphi(a,b)$ and stp(a/A)=stp(a'/A), then follows $\models\varphi(a',b)$).
3. φ is invariant under Lascar strong types over A.
4. For any A-indiscernible sequence $(a_i : i<\omega)$, $\models\psi_\varphi(a_i,a_j)$ holds for all $i<j<\omega$.
5. For all tuples a, $\psi_\varphi(x,a)$ does not divide over A.
6. $D(x=x,\psi_\varphi,k)=D(x=x,\psi_\varphi)=0$.

If T is simple, then the following condition is also equivalent to the previous conditions 1.-6.:

7. For all tuples a, there exists a Morley sequence in tp(a/A) such that $\{\psi_\varphi(x,a_i) : i<\omega\}$ is consistent.

Proof :

1.→2.: Suppose $\models\varphi(a,b)$ and stp(a/A)=stp(a'/A). Then $\models E(a,a')$ for all finite A-definable equivalence relations, in particular $\models\psi_\varphi(a,a')$, by hypothesis. This implies $\models\varphi(a',b)$, since $\models\varphi(a,b)$ holds.

2.→3.: Is clear, since equality of Lascar strong types implies equality of strong types.

3.→1.: It is sufficient to show that $E_L^A(x,y)\vdash\psi_\varphi(x,y)$. Then ψ_φ must be finite, since E_L^A is a bounded equivalence relation. So suppose $\models E_L^A(a,b)$. By Proposition 2.8.20, Lstp(a/A)=Lstp(b/A). As φ is reflexive, $\models\varphi(a,a)$, and from the hypothesis (invariance of φ under Lascar strong types) follows $\models\varphi(a,b)$.

1.→4.: If there was an A-indiscernible sequence $(a_i : i<\omega)$ with $\models\neg\psi_\varphi(a_i,a_j)$ for $i<j<\omega$, then ψ_φ would not be finite.

4.→1.: Suppose ψ_φ is not finite. By compactness, we can find a sequence $(a_i : i<\lambda)$ of arbitrary length λ such that $\models\neg\psi_\varphi(a_i,a_j)$ for all $i<j<\lambda$. So by Proposition 2.2.5, there exists an A-indiscernible sequence $(b_i : i<\omega)$ with $\models\neg\psi_\varphi(b_i,b_j)$ for all $i<j<\omega$. This contradicts 4.

4.→5.: Let a be some tuple, and $(a_i : i<\omega)$ any A-indiscernible sequence in tp(a/A) with a_0=a. By hypothesis, we have $\models\psi_\varphi(a,a_i)$ for all $i<\omega$, whence $\psi_\varphi(x,a)$ does not divide over A.
5.→4.: Suppose there is an A-indiscernible sequence $(a_i : i<\omega)$ such that $\models\neg\psi_\varphi(a_i,a_j)$ for all $i<j<\omega$. Then $\{\psi_\varphi(x,a_i) : i<\omega)$ must be inconsistent, since ψ_φ defines an equivalence relation. Whence, $\psi_\varphi(x,a_0)$ divides over A.
5.↔6. is clear by the Definitions of these ranks.
If T is simple, then the equivalence 5.↔7. follows from Theorem 2.5.10.
q.e.d.

Remark 2.11.43 : Note that in the preceding two Propositions we do not assume simplicity. So these results serve as an approach to the Lstp=stp problem in a more general context.

Corollary 2.11.44 : Let T be simple. If for every 2-thick formula φ the formula ψ_φ is finite, then holds Lstp=stp.

Proof : If T is simple, then $E_L=E_{KP}$. This follows from type-definablity of equality of Lascar strong types (see chapter 2.8). Now, the assertion follws from Corollary 2.8.44 and Theorem 2.11.41.
q.e.d.

Problem 2.11.45 : Let T be simple (short, supershort). Under which conditions one of the equivalent statements in Proposition 2.11.42 holds?

Finally, we would like to present Buechler's proof (in a slightly modified form, following ideas from [Wag]) that low theories satisfy Lstp=stp. Moreover, we shall see, that this proof works for all simple theories in which dividing is type-definable. Whence, as a Corollary of Buechler' proof, we get that if T is simple, and dividing is type-definable (in T), then Lstp=stp holds in T.

Theorem 2.11.46 : Let T be a low theory. Then Lascar strong type is the same as strong type, over any set A.

Proof : Let A be any parameter set. First, we note that equality of Lascar strong types is type definable in simple theories (Corollary 2.8.42), say by the type E(x,y) (to simplify matters, we do not distinguish between the type and the relation defined by this type). Clearly, equality of Lascar strong types implies equality of strong types. So suppose stp(x/A)=stp(y/A). By Proposition 2.10.3, it is sufficient to show that the type-definable equivalence relation E(x,y) describing equality of Lascar strong types of x and y over A is in fact the intersection of a set of definable equivalence relations.

It is clear, that we may restrict our considerations to a complete type $p(x)\in S(A)$, since $stp(x/A)\models tp(x/A)$. So fix a complete type $p(x)\in S(A)$, and consider a symmetric formula $\varphi(x,y)\in E(x,y)$. Let $\Sigma_\varphi(a,b)$ be the following condition:

For all $a'\models Lstp(a/A)$ and all $b'\models Lstp(b/A)$ with $a'\perp_A b'$, the formula $\varphi(x,a')\wedge\varphi(x,b')$ does not fork over A.

Observation: By Proposition 2.8.27, 2.9.16 and 2.9.17, we may assume that $\varphi(x,y)$ is ω-thick, whence $\varphi\in nc_A(x,y)$. Then from Proposition 2.8.42 follows that $\Sigma_\varphi(a,a)$.

Claim: $\Sigma_\varphi(a,b)$ is type-definable over A on realizations a, b of p.

Proof of the Claim: First, note that the condition "$\varphi(x,a')\wedge\varphi(x,b')$ does not fork over A" is invariant under A-automorphisms. So we may assume, without loss of generality, that $a'\perp_A a$ and $b'\perp_A b$ in the definition of the condition Σ_φ. Then we may apply Lemma 2.6.6 and Theorems 2.6.8 and 2.6.15 to see that the formula $\varphi(x,a')\wedge\varphi(x,b')$ is consistent and does not fork over A for *all* independent realizations $a'\models Lstp(a/A)$ and $b'\models Lstp(b/A)$ if and only if this holds for *some* such a', b'. Hence $\Sigma_\varphi(a,b)$ is type-defined by

there exists an A-independent, A-indiscernible sequence $(a_ib_i : i<\omega)$ in $Lstp(a/A)\cup Lstp(b/A)$ such that $a_0\perp_A b_0$ and $\bigwedge_{i<\omega}[\varphi(x,a_i)\wedge\varphi(x,b_i)]$ is consistent,

where the independence can be expressed by $D(a_i/A\cup\{a_jb_j : j<i\},\psi,k)\geq D(p,\psi,k)$ and $D(b_i/A\cup\{a_i,a_jb_j : j<i\},\psi,k)\geq D(p,\psi,k)$ for all $i<\omega$, all formulas ψ, and all $k<\omega$ (see Remark 2.4.3 and Proposition 2.4.14). Q.e.d. Claim.

By Corollary 2.11.24, the negation of Σ_φ is type-definable.

Claim: There is a formula ϑ, such that for all tuples a, b $\models$p the following holds: $\Sigma_\varphi(a,b) \leftrightarrow \models\vartheta(a,b)$.

Proof of the Claim: Suppose q type-defines Σ_φ, and $\neg$q type-defines the negation of Σ_φ. Then $q(x,y)\wedge\neg q(x,y)$ is inconsistent, whence, by compactness, there are formulas $\psi_1\in q$, $\psi_2\in\neg q$ such that $\psi_1\wedge\psi_2$ is inconsistent. But $q(x,y)\vee\neg q(x,y)$ is always true. Then it is easy to see that $(\psi_1\wedge\neg\psi_2)(x,y)$ is equivalent to q(x,y), whence it defines Σ_φ.

q.e.d. Claim.

Clearly, Σ_φ is invariant under Lascar strong type. Hence $p(x)\wedge p(y)\wedge E(x,x')\wedge\vartheta(x,y)\vdash\vartheta(x',y)$; by compactness there is a formula $\psi(x)\in p$ such that $\psi(x)\wedge\psi(y)\wedge E(x,x')\wedge\vartheta(x,y)\vdash\vartheta(x',y)$. Hence

$E_\varphi{}^p(x,x'):=[\psi(x)\leftrightarrow\psi(x')]\wedge\{\psi(x)\rightarrow\vartheta(x,y)\leftrightarrow\vartheta(x',y))]\}$

defines an equivalence relation which is coarser than E. So it has only finitely many classes.

Now suppose a, b $\models$p and $\models E_\varphi{}^p(a,b)$ for all $\varphi\in E$. Choose any $\varphi'(x,y)\in E(x,y)$. As E is an equivalence relation, there is a symmetric formula $\varphi\in E$ such that

$\exists z,z',z''[\varphi(x,z)\wedge\varphi(z,z')\wedge\varphi(z',z'')\wedge\varphi(z'',y)]\vdash\varphi'(x,y)$.

As $E_\varphi{}^p(a,b)$ holds and trivially $\models\psi(a)\wedge\vartheta(a,a)$, we get $\models\vartheta(b,a)$. So there are a', b' with E(a,a') and E(b,b'), such that $\varphi(x,a')\wedge\varphi(x,b')$ contains an element c. As $E(x,y)\vdash\varphi(x,y)$, the choice of φ yields $\models\varphi'(a,b)$; as $\varphi'\in E$ was arbitrary, we get E(a,b).

It follows that E is the conjunction of the definable equivalence relations $E_\varphi{}^p$, for $p\in S(A)$ and $\varphi\in E$.

q.e.d.

Remark 2.11.47 : Clearly, in the definition of the condition Σ_φ we need only the consistency of $\varphi(x,a')\wedge\varphi(x,b')$. But the type-definability of Σ_φ requires the stronger condition of non-forking over A. The invariance under Lascar strong type of the condition Σ_φ plays also a role for the type-definability; furthermore, it guarantees that the equivalence relations $E_\varphi{}^p$ are finite.

2.12 Problems and future studies

The Lstp=stp Problem remains open for (super-) short theories and simple theories in general. Our approachment to the problem by the results 2.11.41, 2.11.42, 2.11.43 could be helpful to solve it.

Theorem 2.11.30 was the starting point to give new characterizations of low theories and has motivated the study of the rank $D(p,\varphi,I)$. It seems to be a useful tool for investigating the relationships between subclasses of simple theories. To demonstrate this we would like to outline the following considerations, which give rise to future works:

Let $\varphi(x,y)$ be some formula and let $J=(a_i^J : i<\alpha_J)$ be a dividing chain in φ of length α_J. We define k_J to be the smallest number such that $\varphi(x,a_0^J)$ k_J-divides over the empty set. Note that we may assume $\alpha_J \geq D(x=x,\varphi,k_J)$. Now let I be any (infinite) independent sequence of $tp(a_0^J)$. By Theorem 2.11.30, the following holds:

$k_J-1 \leq D(x=x,\varphi,I) \leq D(x=x,\varphi,k_J) \leq \alpha_J \leq D(x=x,\varphi)$.

The rank $D(x=x,\varphi,k_J)$ has finite value, since T is simple. If T is short, then α_J is finite; and if T is low, then $D(x=x,\varphi)$ is finite. In these cases it is interesting to ask whether for all formulas $\varphi(x,y)$ and all dividing chains J in φ of length $\alpha_J \geq D(x=x,\varphi,k_J)$, k_J-1 equals $D(x=x,\varphi,k_J)$, α_J, or $D(x=x,\varphi)$, respectively. If the answer was positive, then would follow that $D(x=x,\varphi,I)=D(x=x,\varphi,k_J)$ for all independent sequences (including Morley sequences) I of $tp(a_0^J)$, where a_0^J is the first element of J. That means that $\{\varphi(x,a) : a \in I\}$ is $D(x=x,\varphi,k_J)$-consistent for every independent (Morley-) sequence I of $tp(a_0^J)$. Whence, by Theorem 2.11.30, in this case $D(x=x,\varphi,k_J)+1$ is the smallest number m such that $\{\varphi(x,a) : a \in I\}$ is m-inconsistent for some (every) independent (Morley-) sequence I of $tp(a_0^J)$.

Furthermore, if T is short (or low), and if for all φ and all dividing chains J in φ, $k_J-1=\alpha_J$ (or $k_J-1=D(x=x,\varphi)$), respectively, then follows that T has idcp, witnessed by the independent sequences I of $tp(a_0^J)$, which are dividing chains in φ of length $D(x=x,\varphi,I)=D(x=x,\varphi,k_J)=\alpha_J$ in the short case, and are dividing chains in φ of length $D(x=x,\varphi,I)=D(x=x,\varphi,k_J)=D(x=x,\varphi)$ in the low case.

Which condition implies $k_J-1=\alpha_J$ (if T is short), or $k_J-1=D(x=x,\varphi)$ (if T is low)?

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